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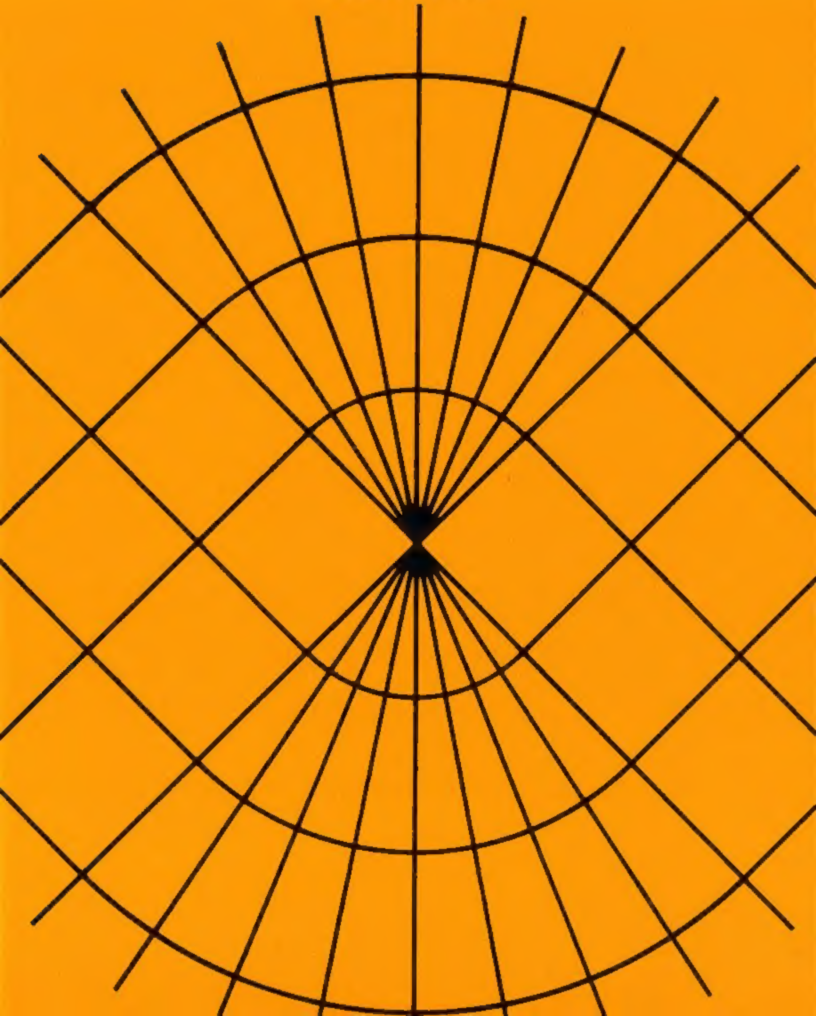
OPTIMUM STRUCTURES

W.S.HEMP



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W. S. HEMP

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1.1 Structures of least volume of material

The data for the present problem consist of (i) given forces applied at given points in space and specified by Cartesian components, (ii) points of support whose forces of constraint are together capable of reacting any given force and couple,† and (iii) a pin-jointed framework of any given layout, whose nodes include the points of application of the given forces and the points of support, and whose members form a structure as opposed to a mechanism. The given forces may be thought of as embracing all components of force (including zero components) acting at all the nodes of the framework, excluding only component forces of constraint. This complete set of given force components is denoted by $F_j (j = 1, 2 \ldots n)$. The points of support must be sufficient to prevent overall motion of the structure, but may be redundant. End loads $T_i (i = 1, 2 \ldots m)$ in the m members of the framework and forces of constraint at the supports can always be found to balance any set of given forces $[F_j]$.‡ An independent set of nodal equilibrium equations for $[T_i]$ can be written

$$\sum_{i=1}^{m} K_{ij} T_i = F_j \qquad (j = 1, 2 \ldots n), \tag{1.1}$$

where $[K_{ij}]$ is determined by the direction cosines of the members of the framework and is therefore known. The order of redundancy of the framework is $m - n$ and should be greater than zero, so as to provide choice for the layout of an optimum structure. The matrix $[K_{ij}]$ will have rank n, since (1.1) is always soluble for n of the T_i.

The framework of (iii) can be given any practical degree of generality by an appropriate choice of its nodes and members. It is completely determined when the areas of cross-section of its members $A_i (i = 1, 2 \ldots m)$ are given, as well as its material of construction. The magnitudes A_i must satisfy

$$A_i \geqslant 0 \qquad (i = 1, 2 \ldots m), \tag{1.2}$$

and the ith member is present if $A_i > 0$ and absent if $A_i = 0$.

† This can always be achieved, even if it is not specified, by adding new supports and perhaps converting some given forces into forces of constraint.
‡ The meaning of the symbols T_i and F_j is illustrated for a particular structure in Fig. 1.1 on p. 7.

1 Pin-jointed frameworks

1.1 Structures of least volume of material

The data for the present problem consist of (i) given forces applied at given points in space and specified by Cartesian components, (ii) points of support whose forces of constraint are together capable of reacting any given force and couple,† and (iii) a pin-jointed framework of any given layout, whose nodes include the points of application of the given forces and the points of support, and whose members form a structure as opposed to a mechanism. The given forces may be thought of as embracing all components of force (including zero components) acting at all the nodes of the framework, excluding only component forces of constraint. This complete set of given force components is denoted by $F_j (j = 1, 2 \ldots n)$. The points of support must be sufficient to prevent overall motion of the structure, but may be redundant. End loads $T_i (i = 1, 2 \ldots m)$ in the m members of the framework and forces of constraint at the supports can always be found to balance any set of given forces $[F_j]$.‡ An independent set of nodal equilibrium equations for $[T_i]$ can be written

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If the structure defined by $[A_i]$ is to transmit the forces $[F_j]$ to the supports with safety, the stresses corresponding to $[T_i]$ must be restricted in magnitude. Let the 'allowable stresses' for a chosen material be σ_T in tension and σ_C in compression. It then follows that

$$T_i \leqslant \sigma_T A_i, \qquad -T_i \leqslant \sigma_C A_i \qquad (i = 1, 2 \ldots m). \tag{1.3}$$

The relations (1.3) must be satisfied even if $[T_i]$ is found by an 'elastic analysis' of the framework. However, for the moment, the simpler procedures of 'plastic analysis' will be followed. They require only that a solution of (1.1) can be found which satisfies (1.3). It will be shown later that this limited approach does in fact lead to an optimum elastic design, which in the case of a single system of given forces coincides with that obtained by plastic or limit analysis.

The optimum problem of the present section can now be formulated. It is to find that framework defined by $[A_i]$, which will safely transmit the given forces $[F_j]$ to the given supports and make the volume V of material in the structure given by

$$V = \sum_{i=1}^{m} A_i l_i, \tag{1.4}$$

where l_i is the length of the ith member, a minimum. The conditions of safety require that non-negative $[A_i]$ be chosen so that (1.1, 3) have a solution $[T_i]$. Mathematically the problem is that of minimizing V of (1.4), subject to the restrictions (1.1, 2, 3).

The relations (1.1–4) can be made non-dimensional by taking $[F_j]$ proportional to F (a typical force), $[T_i]$ to F, σ_T and σ_C to σ (mean of σ_T, σ_C), $[A_i]$ to F/σ, $\{l_i\}$ to l (a typical length) and V to Fl/σ. This means that the best material is that with the largest σ. If a structure of least weight is required, then ρV must be a minimum, where ρ is the specific weight, and so the best material is that with the greatest 'specific stress' σ/ρ.

The relations (1.3) may be replaced by

$$T_i + 2T_i'' = \sigma_T A_i, \quad -T_i + 2T_i' = \sigma_C A_i, \quad T_i' \geqslant 0, \quad T_i'' \geqslant 0 \quad (i = 1, 2 \ldots m), \tag{1.5}$$

where $[T_i']$, $[T_i'']$ are 'slack variables', and the first two of (1.5) solved to give

$$A_i = (T_i' + T_i'')/\sigma, \qquad T_i = (\sigma_T T_i' - \sigma_C T_i'')/\sigma \qquad (i = 1, 2 \ldots m), \tag{1.6}$$

where,

$$\sigma = (\sigma_T + \sigma_C)/2. \tag{1.7}$$

Substituting from (1.6) into (1.1) and (1.4) then gives a new formulation of the present problem in the form:

$$\min V = \sum_{i=1}^{m} (T_i' + T_i'')l_i/\sigma, \tag{1.8}$$

subject to

$$T_i', \ T_i'' \geqslant 0 \qquad (i = 1, 2 \ldots m), \tag{1.9}$$

and

$$\sum_{i=1}^{m} K_{ij}(\sigma_T T_i' - \sigma_C T_i'')/\sigma = F_j \qquad (j = 1, 2 \ldots n). \tag{1.10}$$

Equations (1.8, 9, 10) define a problem of linear programming.

1.2 The simplex method

The problem of (1.8–10) is a special case of the standard problem of linear programming:

$$\min z = \sum_{i=1}^{p} c_i x_i, \tag{1.11}$$

subject to

$$x_i \geqslant 0 \qquad (i = 1, 2 \ldots p), \tag{1.12}$$

and

$$\sum_{i=1}^{p} a_{ij} x_i = b_j \qquad (j = 1, 2 \ldots n), \tag{1.13}$$

where $\{c_i\}$, $[a_{ij}]$ and $[b_j]$ are given matrices. The linear function z of (1.11) is called the 'objective function'. The relations (1.12) are termed the 'non-negativity restrictions' and the equations of (1.13) the 'constraints'. Non-negative values of the variables $[x_i]$ are to be found which satisfy the constraints and make the objective function a minimum.

It is assumed that $[a_{ij}]$ has rank n and so (1.13) can be solved, after a renumbering of the variables perhaps, in the form

$$x_j = x_{Bj} - \sum_{i=n+1}^{p} y_{ij} x_i \qquad (j = 1, 2 \ldots n). \tag{1.14}$$

This implies that

$$x_j = x_{Bj} \quad (j = 1, 2 \ldots n), \qquad x_i = 0 \quad (i = n+1, \ldots p) \tag{1.15}$$

is a solution of (1.13). It is termed a 'basic solution', meaning that it contains at most n non-zero components. The variables $x_j (j = 1, 2 \ldots n)$ are termed 'basic

variables' and $x_i (i = n+1, .. p)$ 'non-basic variables'. Any solution like (1.14) divides $[x_i]$ into variables of these two kinds.

Assume now that the constants $x_{Bj} (j = 1, 2 .. n)$ satisfy

$$x_{Bj} \geqslant 0 \qquad (j = 1, 2 .. n). \tag{1.16}$$

The solution (1.15) then satisfies (1.12) and is for this reason termed 'feasible'. It is thus a 'basic feasible solution'. It is always possible to find such solutions of (1.10), since any solution of (1.10) can be taken as giving for each of n values of i either T_i' or $-T_i''$.

Equation (1.14) is equivalent to (1.13) and gives feasible solutions, with $x_i \geqslant 0$ $(i = n+1, .. p)$, so long as $x_j \geqslant 0$ $(j = 1, 2 .. n)$ is satisfied. Substitution from (1.14) into (1.11) gives

$$z = z_B + \sum_{i=n+1}^{p} (c_i - z_i) x_i, \tag{1.17}$$

with

$$z_B = \sum_{j=1}^{n} c_j x_{Bj}, \tag{1.18}$$

and

$$z_i = \sum_{j=1}^{n} c_j y_{ij} \qquad (i = n+1, .. p). \tag{1.19}$$

The quantity z_B is the value of the objective function corresponding to the basic solution (1.15). The quantities z_i $(i = n+1, .. p)$ can be obtained once a solution (1.14) has been found and are thus defined for any basic solution.

Sufficient conditions for (1.15) to give a minimum of z equal to z_B are now seen by (1.17) to be

$$c_i \geqslant z_i \qquad (i = n+1, .. p), \tag{1.20}$$

since, if (1.20) is satisfied, no non-negative x_i $(i = n+1, .. p)$ can decrease z in (1.17) below z_B.

If (1.20) is false for at least one of $i = n+1, .. p$, say $i = k$, that is if

$$c_k < z_k, \tag{1.21}$$

then it is possible by giving x_k positive values and setting $x_i = 0$ $(i \neq k)$ to decrease z below z_B. The variables x_j of (1.14) are then given by

$$x_j = x_{Bj} - y_{kj} x_k \geqslant 0 \qquad (j = 1, 2 .. n), \tag{1.22}$$

and must satisfy the inequality imposed. This means that

$$x_k \leqslant x_{Bj}/y_{kj}, \qquad \text{for all } j \text{ with } y_{kj} > 0. \tag{1.23}$$

There is no restriction on x_k if $y_{kj} \leqslant 0$ $(j = 1, 2 .. n)$ and in this case, by (1.17), $z \to -\infty$ as $x_k \to +\infty$. Such 'unbounded solutions' do not arise in structural applications; V in (1.8) cannot be negative. Assume therefore that there is a value of r taken from $j = 1, 2 .. n$ such that

$$x_{Br}/y_{kr} = \min_j x_{Bj}/y_{kj}, \qquad \text{for } y_{kj} > 0. \tag{1.24}$$

The assumption that $x_k = x_{Br}/y_{kr}$ then satisfies all of (1.22) and the solution

$$\left.\begin{aligned} &x_j = x_{Bj} - y_{kj} x_{Br}/y_{kr} \qquad (j = 1, 2 .. n;\ j \neq r), \\ &x_k = x_{Br}/y_{kr}, \\ &x_j = 0 \qquad (j = n+1, .. p;\ j \neq k), \\ &x_r = 0, \end{aligned}\right\} \tag{1.25}$$

in which the previously basic variable x_r has been interchanged with the previously non-basic variable x_k, is a new basic feasible solution. The corresponding value of z is by (1.17)

$$z = z_B + (c_k - z_k) x_{Br}/y_{kr}, \tag{1.26}$$

which thanks to (1.21) gives a smaller value of z than z_B, unless $x_{Br} = 0$.

A basic solution with one or more of its basic variables equal to zero is said to be 'degenerate'. If (1.15) is degenerate and $x_{Bj} = 0$ for a value of j for which $y_{kj} > 0$, then by (1.24) $x_{Br} = 0$ and no decrease of z is possible by inserting x_k among the basic variables. However z is not increased by this insertion and a new set of basic variables x_j $(j = 1, 2 .. n;\ j \neq r)$, x_k has been introduced.

It is to be noticed that ambiguity in the choice of r in (1.24) leads to degeneracy in the new basic solution of (1.25), since at least one of the equations in the first line of (1.25) will give a zero value.

The process of interchanging basic and non-basic variables described above is called a 'simplex transformation' or 'simplex iteration'. It can be continued as long as a k can be found, which is the index of a non-basic variable for which (1.21) is valid. It will eventually determine an 'optimum solution', for which z is a minimum, or an unbounded solution, because these are the only ways it can stop and the number of basic solutions like (1.15) are finite in number. A possible difficulty is 'cycling', which can occur if a basic solution repeats itself during the iterative process. This can only happen if there is degeneracy, because otherwise z is decreased at each iteration. It can be overcome by an appropriate selection of basic and non-basic variables for an interchange, in cases where there is more than one possibility. Fortunately cycling arises but rarely in practical calculations. Means for avoiding it are given in the standard works on linear programming (Dantzig 1963; Hadley 1965).

Equations corresponding to (1.14), for the new basic variables of the solution of (1.25), can be obtained by solving the rth equation of (1.14) for x_k

and substituting in the remaining equations. This gives

$$\left.\begin{aligned} x_j &= (x_{Bj} - y_{kj}\, x_{Br}/y_{kr}) - \left\{ \sum_{\substack{i=n+1 \\ i \neq k}}^{p} (y_{ij} - y_{kj} y_{ir}/y_{kr}) x_i + (-y_{kj}/y_{kr}) x_r \right\} \\ &\qquad\qquad\qquad\qquad\qquad (j = 1, 2 \,..\, n;\ j \neq r), \\ x_k &= x_{Br}/y_{kr} - \left\{ \sum_{\substack{i=n+1 \\ i \neq k}}^{p} (y_{ir}/y_{kr}) x_i + x_r/y_{kr} \right\}, \end{aligned}\right\} \tag{1.27}$$

which can be used like (1.14) to carry out a new iteration if required.

The choice of k is governed by (1.21). If there is ambiguity, then (1.26) suggests that the most rapid approach to the minimum of z is obtained by taking that k, which makes

$$(z_k - c_k)\, x_{Br}/y_{kr} \quad \text{a maximum.} \tag{1.28}$$

In practical calculations the simpler criterion

$$z_k - c_k \quad \text{is a maximum} \tag{1.29}$$

is usually used.

The process of iteration described above is called the 'simplex method' of resolving problems of linear programming. It is due to G. B. Dantzig and is readily programmed for the digital computer. Many standard programmes are available, see for example Kienzi, Tzschach, and Zehnder (1968), and so there is no difficulty in solving quite complicated problems of the type defined by (1.8–10). The only limit is available storage.

It has already been remarked that (1.10) may be solved in the form (1.14) with (1.16) satisfied. Any statically determinate sub-framework, preferably one which experience suggests is a 'good design', may be used to isolate n quantities T_i, including the loads in this sub-framework, and then (1.1) may be solved for these variables in terms of the given forces $[F_j]$ and the remaining $m - n$ quantities T_i. This defines a solution to (1.10) as well, since this is the same set of equations, and leads to a division of T_i', T_i'' $(i = 1, 2 \,..\, m)$ into basic and non-basic variables, with an associated basic feasible solution, because a non-negative term in this solution can always be obtained by choosing one of T_i' or T_i'' as the basic variable. It is to be remarked that if one of T_i', T_i'' is a basic variable, then the other must be non-basic and so, for all i, one of T_i', T_i'' is always zero in a basic solution. Both can be zero of course, if T_i' and T_i'' are non-basic variables.

The simplex method leads to a definite minimum value for V, since, as has already been remarked, unbounded solutions are not possible because $V \geqslant 0$. The basic solution corresponding to $V_{\min}$ defines an optimum framework and so demonstrates the existence of such a framework of least volume of material designed according to the methods of plastic design. This optimum framework is statically determinate and so is an elastic design as well as a plastic one. Now

since elastic designs have to satisfy the additional conditions of compatibility of strains, the minimum value of V for elastic designs cannot be less than that for plastic designs determined here. However the optimum plastic design is an elastic design too and so its value of V is both $\leqslant$ and $\geqslant$ the minimum V for elastic designs. The two minima thus coincide and the least volume for elastic designs is equal to that for plastic designs.

1.3 An example

Consider the symmetrically loaded symmetrical framework of Fig. 1.1. The figure shows the most general possible system of given forces and a complete

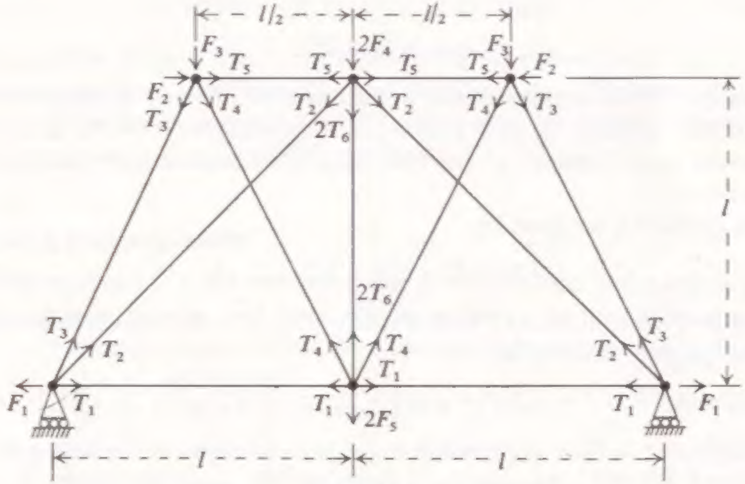


Fig. 1.1

system of end loads in the members. An optimum design is sought for the special loading system

$$F_1 = F_2 = F_3 = F_4 = 0, \quad F_5 = F. \tag{1.30}$$

The equations of nodal equilibrium corresponding to (1.1) are

$$\left.\begin{aligned} T_1 + T_2/\sqrt{2} + T_3/\sqrt{5} &= F_1 = 0, \\ T_3/\sqrt{5} - T_4/\sqrt{5} - T_5 &= F_2 = 0, \\ -2T_3/\sqrt{5} - 2T_4/\sqrt{5} &= F_3 = 0, \\ -T_2/\sqrt{2} - T_6 &= F_4 = 0, \\ 2T_4/\sqrt{5} + T_6 &= F_5 = F, \end{aligned}\right\} \tag{1.31}$$

and have a solution

$$T_1 = F - T_4/\sqrt{5}, \quad T_2 = -\sqrt{2}F + 2\sqrt{(2/5)}\,T_4, \quad T_3 = -T_4, \quad T_5 = -2T_4/\sqrt{5},$$
$$T_6 = F - 2T_4/\sqrt{5}. \tag{1.32}$$

Assume for simplicity that $\sigma_T = \sigma_C = \sigma$. Equation (1.6) then gives

$$A_i = (T_i' + T_i'')/\sigma, \quad T_i = T_i' - T_i'' \qquad (i = 1, 2 \,..\, 6). \tag{1.33}$$

Equations (1.32) then yield, using (1.33),

$$\left.\begin{aligned} T_1' &= F + T_1'' - (T_4' - T_4'')/\sqrt{5} = \sigma A_1 - T_1'' \\ T_2'' &= \sqrt{2}F + T_2' - 2\sqrt{(2/5)}(T_4' - T_4'') = \sigma A_2 - T_2', \\ T_3'' &= T_3' + T_4' - T_4'' = \sigma A_3 - T_3', \\ T_5'' &= 2(T_4' - T_4'')/\sqrt{5} + T_5' = \sigma A_5 - T_5', \\ T_6' &= F - 2(T_4' - T_4'')/\sqrt{5} + T_6'' = \sigma A_6 - T_6''. \end{aligned}\right\} \tag{1.34}$$

Also,

$$\sigma A_4 = T_4' + T_4''. \tag{1.35}$$

The choice of basic variables is clear in the case of T_1', T_2'' and T_6', since these take positive values in the basic solution corresponding to (1.34). T_3'' and T_5'' are chosen, since if members 3 and 5 do come into operation they will be in compression.

The values of l_i are given by

$$l_i = l(1, \sqrt{2}, \sqrt{5}/2, \sqrt{5}/2, 1/2, 1). \tag{1.36}$$

Substituting from (1.34), (1.35), (1.36) into (1.4) or (1.8) then determines V for the complete structure as

$$V = (2l/\sigma)(4F + 2T_1'' + 2\sqrt{2}T_2' + \sqrt{5}T_3' - T_4'/\sqrt{5} + 6T_4''/\sqrt{5} + T_5' + 2T_6''). \tag{1.37}$$

The coefficient of T_4' is negative and so the basic solution corresponding to (1.34) with $V = 8lF/\sigma$ does not give a minimum. The non-basic variable T_4' must therefore be exchanged for a basic variable. The rule of (1.24) gives T_2'' or T_6' for the exchange. Taking T_2'' and solving the second of (1.34) for T_4' gives

$$T_4' = \sqrt{5}F/2 + \sqrt{(5/2)}(T_2' - T_2'')/2 + T_4'', \tag{1.38}$$

which when substituted in the remaining equations of (1.34) and in (1.37) gives

$$\left.\begin{aligned} T_1' &= F/2 + T_1'' - (T_2' - T_2'')/2\sqrt{2}, \\ T_3'' &= \sqrt{5}F/2 + \sqrt{(5/2)}(T_2' - T_2'')/2 + T_3', \\ T_5'' &= F + (T_2' - T_2'')/\sqrt{2} + T_5', \\ T_6' &= -(T_2' - T_2'')/\sqrt{2} + T_6'', \end{aligned}\right\} \tag{1.39}$$

and

$$V = (2l/\sigma)(7F/2 + 2T_1'' + 7T_2'/2\sqrt{2} + T_2''/2\sqrt{2} + \sqrt{5}T_3' + \sqrt{5}T_4'' + T_5' + 2T_6''). \tag{1.40}$$

The new basic solution gives an optimum, since V in (1.40) cannot be decreased below $7lF/\sigma$. The minimum V is thus

$$V_{\min} = 7lF/\sigma, \tag{1.41}$$ †

with

$$T_1'' = T_2' = T_2'' = T_3' = T_4'' = T_5' = T_6'' = 0. \tag{1.42}$$

Substituting from (1.42) into (1.38, 39) and using (1.33) then gives

$$T_i = F(1/2, 0, -\sqrt{5}/2, \sqrt{5}/2, -1, 0), \tag{1.43}$$

and

$$A_i = (F/\sigma)(1/2, 0, \sqrt{5}/2, \sqrt{5}/2, 1, 0). \tag{1.44}$$

The optimum design is thus determined. The layout follows from Fig. 1.1, where now members 2 and 6 should be omitted. The node at the force $2F_4$ should also be removed to avoid instability. The present theory does not require this, since it takes no account of such matters.

1.4 Duality in linear programming

The problem of (1.11–13) is now called the 'primal problem' and is assumed to have an optimum solution given by (1.15). Equations (1.13) are soluble for x_i $(i = 1, 2 \,..\, n)$ and this process may be made explicit by introducing co-factors A_{jk} $(j, k = 1, 2 \,..\, n)$, which satisfy

$$\sum_{j=1}^{n} a_{ij}A_{jk} = \begin{cases} 1 \; (i = k) \\ 0 \; (i \neq k) \end{cases} \qquad (i, k = 1, 2 \,..\, n). \tag{1.45}$$

The constants of (1.14) can then be written

$$x_{Bk} = \sum_{j=1}^{n} b_j A_{jk} \qquad (k = 1, 2 \,..\, n), \tag{1.46}$$

and

$$y_{ik} = \sum_{j=1}^{n} a_{ij}A_{jk} \qquad (i = n+1, \,..\, p; \; k = 1, 2 \,..\, n). \tag{1.47}$$

The minimum value z_B of (1.18) can be expressed, using (1.46), as

$$z_{\min} = z_B = \sum_{k=1}^{n} \sum_{j=1}^{n} b_j A_{jk} c_k \tag{1.48}$$

† This may be compared with the absolute optimum for the half-plane, which by (4.56) has $V_{\min} = 2\pi lF/\sigma$.

and the condition (1.20), using (1.19) and (1.47), can be written

$$\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ij}A_{jk}c_k \leqslant c_i \qquad (i = n+1, \ldots p). \tag{1.49}$$

The 'dual problem' or the 'dual' of (1.11–13) can be written as:

$$\max w = \sum_{j=1}^{n} b_j u_j, \tag{1.50}$$

subject to,

$$\sum_{j=1}^{n} a_{ij}u_j \leqslant c_i \qquad (i = 1, 2 \ldots p). \tag{1.51}$$

The relation (1.51) can also be written as

$$\sum_{j=1}^{n} a_{ij}u_j + v_i = c_i, \quad v_i \geqslant 0 \qquad (i = 1, 2 \ldots p). \tag{1.52}$$

The dual problem depends on the same constants as the primal, but in a different way. The dual variables $\{u_j\}$, which can have any sign, are n in number, and this by (1.13) is the number of constraints of the primal. The number of constraints in the dual, which by (1.51) is p, is equal to the number of variables $[x_i]$ in the primal.

Substitution from (1.13) into (1.50) and use of (1.52) and (1.11) gives

$$w = \sum_{j=1}^{n} b_j u_j = \sum_{i=1}^{p}\sum_{j=1}^{n} x_i a_{ij} u_j = \sum_{i=1}^{p} (c_i x_i - x_i v_i) = z - \sum_{i=1}^{p} x_i v_i, \tag{1.53}$$

and so, since by (1.12), (1.52) x_i, v_i $(i = 1, 2 \ldots p)$ are all non-negative,

$$w = \sum_{j=1}^{n} b_j u_j \leqslant z_{\min} \leqslant \sum_{i=1}^{p} c_i x_i = z, \tag{1.54}$$

where $\{u_j\}$ is a solution of the dual constraint (1.51) and $[x_i]$ is a feasible solution of the primal constraint (1.13). The relations (1.54) give a means of finding upper and lower bounds to $z_{\min}$. They also show that w is bounded above.

It can now be shown that the dual has an optimum solution given by

$$u_j = \sum_{k=1}^{n} A_{jk} c_k \qquad (j = 1, 2 \ldots n). \tag{1.55}$$

Substitution in (1.52) gives

$$\sum_{j=1}^{n}\sum_{k=1}^{n} a_{ij}A_{jk}c_k + v_i = c_i \qquad (i = 1, 2 \ldots p), \tag{1.56}$$

and so by (1.45) and (1.49)

$$v_i = 0 \quad (i = 1, 2 \ldots n); \qquad v_i \geqslant 0 \quad (i = n+1, \ldots p), \tag{1.57}$$

which shows that (1.55) satisfies the dual constraints (1.52) or (1.51). The value of w corresponding to (1.55) is, using (1.48), given by

$$w = \sum_{j=1}^{n}\sum_{k=1}^{n} b_j A_{jk} c_k = z_{\min}, \tag{1.58}$$

and so by (1.54) it follows that the dual has a solution and that

$$w_{\max} = z_{\min}. \tag{1.59}$$

The value of $z_{\min}$ can thus be obtained by solving the dual problem.

Equation (1.53) gives for the optimum solutions to the primal and the dual, since (1.59) is valid for them, $\sum_{i=1}^{p} x_i v_i = 0$ or since each term of this sum is positive,

$$x_i v_i = 0 \qquad (i = 1, 2 \ldots p). \tag{1.60}$$

This relation is verified by (1.15) and (1.57). It shows that if an optimum solution of the primal has a component $x_i > 0$, then the corresponding constraint of the dual is satisfied as an equality in its optimum solution. It shows further that the variable of the primal x_i, which corresponds to a constraint of the dual satisfied in its optimum solution as a strict inequality, must vanish in the optimum solution.

The conditions (1.60) are thus necessary for optimum solutions. They are also sufficient, since, by (1.53), they imply that the w and z, corresponding to a solution of the dual constraints and a feasible solution of the primal constraints, are equal to one another and so by (1.54, 59) equal to their common optimum values. This result can also be expressed as the following theorem.

The necessary and sufficient condition that a feasible solution $[x_i]$ shall be optimum for the primal (1.11–13), which is known to have an optimum, is that there should be a solution $\{u_j\}$ to the dual constraints (1.51), which satisfies these constraints as an equality for all values of i for which $x_i > 0$. (1.61)

The dual problem was introduced in (1.50, 51) by definition. It can also be approached by introducing 'Lagrangian multipliers' $\{u_j\}$ in an attempt to solve the primal problem. The 'Lagrangian' Z corresponding to (1.11, 13) is

$$\begin{aligned} Z &= \sum_{i=1}^{p} c_i x_i + \sum_{j=1}^{n} u_j \left(b_j - \sum_{i=1}^{p} a_{ij} x_i\right) \\ &= \sum_{j=1}^{n} b_j u_j + \sum_{i=1}^{p} \left(c_i - \sum_{j=1}^{n} a_{ij} u_j\right) x_i, \end{aligned} \tag{1.62}$$

and the necessary conditions that this function of non-negative $[x_i]$ should have a minimum are that (1.51) should be valid, since otherwise $x_i \to \infty$ (any i with (1.51) false) implies $Z \to -\infty$, and that

$$\sum_{j=1}^{n} a_{ij} u_j = c_i \qquad (x_i > 0). \tag{1.63}$$

These are in fact the conditions of (1.61), which are thus expressed in a Lagrangian form.

If Z is now thought of as function of non-negative $[x_i]$ and $\{u_j\}$, then the condition that it is stationary with respect to $\{u_j\}$ is the constraint (1.13). If this last is imposed upon the problem 'make Z stationary', Z reduces to z and the primal problem is obtained, which is taken here to be a minimum problem. If on the other hand, the conditions of (1.51) and (1.63) are imposed, then Z reduces to w a function of $\{u_j\}$ and (1.51, 63) with $[x_i]$ eliminated, reduce to (1.51). This gives the dual problem, which is known to be a maximum problem. The primal and dual problems can thus be generated in a reciprocal way from the necessary conditions that the Lagrangian Z should be stationary with regard to $\{u_j\}$ and a minimum with respect to $[x_i]$.

1.5 The dual problem for structures of least volume of material

The primal problem for structures of least volume of material is given by (1.8–10). Comparison with (1.11–13) gives

$$\left.\begin{aligned}
[x_i] &= [T_1', \dots T_m', T_1'', \dots T_m''],\\
\{c_i\} &= \{l_1, \dots l_m, l_1, \dots l_m\}/\sigma,\\
[a_{ij}] &= [\sigma_T K_{1j}/\sigma, \dots \sigma_T K_{mj}/\sigma, -\sigma_C K_{1j}/\sigma, \dots -\sigma_C K_{mj}/\sigma],\\
[b_j] &= [F_j].
\end{aligned}\right\} \tag{1.64}$$

The dual problem can thus be written using (1.50, 51) as

$$\max W = \sum_{j=1}^{n} F_j u_j/\sigma\epsilon, \tag{1.65}$$

subject to,

$$\sum_{j=1}^{n} K_{ij} u_j \leqslant (\sigma\epsilon/\sigma_T) l_i, \quad -\sum_{j=1}^{n} K_{ij} u_j \leqslant (\sigma\epsilon/\sigma_C) l_i \qquad (i = 1, 2 \dots m), \tag{1.66}$$

where the dual variables are $u_j/\sigma\epsilon$, with σ given by (1.7) and ϵ a positive infinitesimal.

Let $\{u_j\}$ be interpreted as virtual displacement components at the nodes corresponding to the given forces $[F_j]$. The objective function W is then the

virtual work of the given forces taken over the virtual displacements, divided by the constant $\sigma\epsilon$. Equation (1.1) gives for any $\{u_j\}$

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{n} K_{ij} u_j \right) T_i = \sum_{j=1}^{n} F_j u_j, \tag{1.67}$$

which is the principle of virtual work. The virtual strains $\{\epsilon_i\}$ in the members of the framework are thus given by

$$\epsilon_i = \sum_{j=1}^{n} K_{ij} u_j / l_i \qquad (i = 1, 2 \dots m), \tag{1.68}$$

and so (1.66) can be written as

$$\epsilon_i \leqslant \sigma\epsilon/\sigma_T, \quad -\epsilon_i \leqslant \sigma\epsilon/\sigma_C \qquad (i = 1, 2 \dots m), \tag{1.69}$$

which implies that the virtual strains must lie between $-\sigma\epsilon/\sigma_C$ and $\sigma\epsilon/\sigma_T$. The dual problem thus seeks to maximize the virtual work W, subject to bounds placed on the values of virtual strains as given by (1.69).

The theorem of (1.61) leads to the following conditions for optimum frameworks:

The necessary and sufficient conditions that a pin-jointed framework, selected from a given framework, should have a minimum volume of material are that it should be capable of carrying its given forces with stresses σ_T in its tension members and $-\sigma_C$ in its compression members and should allow a virtual displacement of its nodes, which produces a strain of $\sigma\epsilon/\sigma_T$ in its tension members and a strain of $-\sigma\epsilon/\sigma_C$ in its compression members and strains not outside this range in members of the given framework, which are not present in the optimum design. (1.70)

The primal variables T_i', T_i'', for any i, cannot both be positive, since this would imply contradictory equalities in (1.69). One or both of T_i', T_i'', for each i, must thus be zero. If $T_i' > 0$, $T_i'' = 0$, then by (1.6) $T_i > 0$ and $T_i/A_i = \sigma_T$. Also by (1.69) $\epsilon_i = \sigma\epsilon/\sigma_T$. If $T_i' = 0$, $T_i'' > 0$, then by (1.6) $T_i < 0$ and $T_i/A_i = -\sigma_C$. Also by (1.69) $\epsilon_i = -\sigma\epsilon/\sigma_i$. If $T_i' = T_i'' = 0$, then by (1.6) $T_i = 0$ and $A_i = 0$. Also (1.69) must be satisfied.

Conditions of the type found in (1.70) were first given in Michell (1904) and are called Michell conditions in his honour. Michell considered sufficient conditions only and dealt with 'continuous' strain fields and structures. An account of his theory is to be found in Chapter 4.

1.6 Direct solution of problems using the dual

The dual problem of (1.65, 66) can be transformed to the standard form of (1.11–13) by introducing non-negative slack variables in (1.66) and replacing $\{u_j\}$ by

$$u_j = u_j' - u_j''; \quad u_j', u_j'' \geqslant 0 \qquad (j = 1, 2 \dots n). \tag{1.71}$$

Relations (1.65, 66) then yield

$$\max W = \sum_{j-1}^{n} F_j(u_j' - u_j'')/\sigma\epsilon, \qquad (1.72)$$

subject to

$$\sum_{j=1}^{n} K_{ij}(u_j' - u_j'') + p_i = (\sigma\epsilon/\sigma_T)l_i, \quad -\sum_{j=1}^{n} K_{ij}(u_j' - u_j'') + q_i = (\sigma\epsilon/\sigma_C)l_i$$

$$(i = 1, 2 \ldots m), \qquad (1.73)$$

and

$$u_j', u_j'' \geqslant 0 \quad (j = 1, 2 \ldots n); \qquad p_i, q_i \geqslant 0 \quad (i = 1, 2 \ldots m). \qquad (1.74)$$

Equations (1.73) may also be written

$$\sum_{j=1}^{n} K_{ij}(u_j' - u_j'') + p_i = (\sigma\epsilon/\sigma_T)l_i, \quad p_i + q_i = \sigma\epsilon l_i(1/\sigma_T + 1/\sigma_C) \quad (i = 1, 2 \ldots m) \qquad (1.75)$$

which shows that $\{p_i\}$ can be treated as 'bounded variables'. This means that analysis by the simplex method can be somewhat shortened (Dantzig 1963, Chap. 18).

Equations corresponding to (1.14) can be obtained from (1.73) in the form

$$\left.\begin{aligned} p_i &= (\sigma\epsilon/\sigma_T)l_i - \sum_{j=1}^{n} K_{ij}(u_j' - u_j'') \\ q_i &= (\sigma\epsilon/\sigma_C)l_i + \sum_{j=1}^{n} K_{ij}(u_j' - u_j'') \end{aligned}\right\} \quad (i = 1, 2 \ldots m), \qquad (1.76)$$

which give a basic feasible solution, from which the simplex method can begin.

The final optimum solution or any basic solution for that matter, cannot contain both u_j' and u_j'', for any j, among the basic variables, since (1.73) could not be solved for such a pair of variables among others, because the corresponding determinant vanishes. The apparent indeterminancy of u_j', u_j'' in (1.71) thus disappears for basic solutions.

The optimum solution will determine the virtual displacements $\{u_j\}$ and also give those members which have $p_i = 0$ or $q_i = 0$, i.e. those which reach the strain limits $\sigma\epsilon/\sigma_T$ and $-\sigma\epsilon/\sigma_C$. This gives a layout of tension and compression members from which the optimum structure can be found. In many cases more potential members will be supplied than are required to form an optimum structure. This surplus may be eliminated by imposing the static conditions (1.1), but sometimes a redundant structure will remain. This defines a class of optimum structures with the same volume of material, which is more general than could be obtained from a basic solution of the primal.

In the special case when

$$\sigma_T = \sigma_C = \sigma \qquad (1.77)$$

the solution to the dual shows that, for the optimum framework, members in tension with stress σ will have a virtual strain ϵ and members in compression with stress $-\sigma$ will have a virtual strain $-\epsilon$. It follows that if

$$\epsilon = \sigma/E, \qquad (1.78)$$

where E is Young's modulus for the material of the framework, then the virtual strains coincide with the real strains in the members and the conditions of compatibility of strain are satisfied. The optimum is thus an optimum elastic design, even though the optimum framework is a redundant structure.

1.7 A further example

Consider the problem defined by Fig. 1.2. This shows the geometry of a given layout for a framework, which is taken as symmetrical about the line through

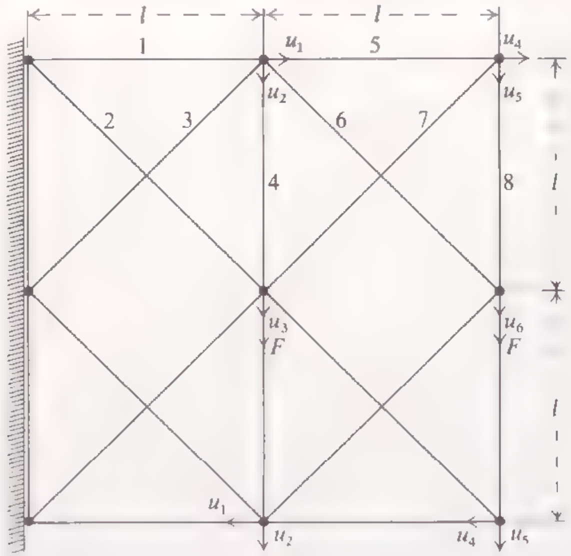


Fig. 1.2

the nodes where the given forces F are acting. The enumeration of the members is shown as are the components of virtual displacements $\{u_j\}$. The virtual strains $\{\epsilon_i\}$ in the members are given by

$$\left.\begin{aligned} &\epsilon_1 = u_1/l, \ \epsilon_2 = u_3/2l, \ \epsilon_3 = (u_1 - u_2)/2l, \ \epsilon_4 = (u_3 - u_2)/l, \\ &\epsilon_5 = (u_4 - u_1)/l, \ \epsilon_6 = (u_6 - u_1 - u_2)/2l, \ \epsilon_7 = (u_4 - u_5 + u_3)/2l, \\ &\qquad\qquad\qquad\qquad \epsilon_8 = (u_6 - u_5)/l. \end{aligned}\right\} \qquad (1.79)$$

Assuming for simplicity that $\sigma_T = \sigma_C = \sigma$, the constraints of (1.69) can be written

$$\epsilon_i \leqslant \epsilon, \quad -\epsilon_i \leqslant \epsilon \qquad (i = 1, 2 \ldots 8) \tag{1.80}$$

and so, by (1.79) introducing slack variables† $\{p_i\}$ and $\{q_i\}$, yield

$$\left.\begin{aligned}
&u_1 + p_1 = \epsilon l, \quad -u_1 + q_1 = \epsilon l,\\
&u_3 + p_2 = 2\epsilon l, \quad -u_3 + q_2 = 2\epsilon l,\\
&u_1 - u_2 + p_3 = 2\epsilon l, \quad -u_1 + u_2 + q_3 = 2\epsilon l,\\
&u_3 - u_2 + p_4 = \epsilon l, \quad -u_3 + u_2 + q_4 = \epsilon l,\\
&u_4 - u_1 + p_5 = \epsilon l, \quad -u_4 + u_1 + q_5 = \epsilon l,\\
&u_6 - u_1 - u_2 + p_6 = 2\epsilon l, \quad -u_6 + u_1 + u_2 + q_6 = 2\epsilon l,\\
&u_4 - u_5 + u_3 + p_7 = 2\epsilon l, \quad -u_4 + u_5 - u_3 + q_7 = 2\epsilon l,\\
&u_6 - u_5 + p_8 = \epsilon l, \quad -u_6 + u_5 + q_8 = \epsilon l,
\end{aligned}\right\} \tag{1.81}$$

with

$$p_i \geqslant 0, \quad q_i \geqslant 0 \qquad (i = 1, 2 \ldots 8). \tag{1.82}$$

The sixteen equations (1.81) can be solved for u_j $(j = 1, 2 \ldots 6)$ and for ten of the slack variables. It is clear in all cases which sign the virtual strain of magnitude ϵ will take, if the corresponding member is present in the optimum structure. This suggests that p_1, p_2, q_3, p_5, p_6 and p_8 could be taken as non-basic variables. Following this suggestion leads to

$$\left.\begin{aligned}
&u_1 = \epsilon l - p_1, \quad q_1 = 2\epsilon l - p_1,\\
&u_3 = 2\epsilon l - p_2, \quad q_2 = 4\epsilon l - p_2,\\
&u_2 = 3\epsilon l - p_1 - q_3, \quad p_3 = 4\epsilon l - q_3,\\
&p_4 = 2\epsilon l - p_1 + p_2 - q_3, \quad q_4 = p_1 - p_2 + q_3,\\
&u_4 = 2\epsilon l - p_1 - p_5, \quad q_5 = 2\epsilon l - p_5,\\
&u_6 = 6\epsilon l - 2p_1 - q_3 - p_6, \quad q_6 = 4\epsilon l - p_6,\\
&u_5 = 5\epsilon l - 2p_1 - q_3 - p_6 + p_8, \quad q_8 = 2\epsilon l - p_8,\\
&p_7 = 3\epsilon l - p_1 + p_2 - q_3 + p_5 - p_6 + p_8,\\
&q_7 = \epsilon l + p_1 - p_2 + q_3 - p_5 + p_6 - p_8,
\end{aligned}\right\} \tag{1.83}$$

which defines a basic feasible solution. There is no need to introduce $\{u_j'\}$ and $\{u_j''\}$ from (1.71) here, since the determination of the maximum depends, thanks to (1.83), on the slack variables only.‡

† It is to be remarked that the quantities differ from those of (1.73) by numerical multipliers.

‡ It is always worthwhile to solve as in (1.83) in simple problems. The result may not always lead to a feasible solution however and certain simplex iterations, involving the slack variables, may be required to achieve feasibility, before the process of maximization can begin.

For the present problem W of (1.65) is given by

$$W = F(u_3 + u_6)/\sigma\epsilon = F(8\epsilon l - 2p_1 - p_2 - q_3 - p_6)/\sigma\epsilon, \tag{1.84}$$

where (1.83) has been used. Thanks to (1.82) the maximum follows without further calculation. It is

$$W_{\max} = 8Fl/\sigma, \tag{1.85}$$

with

$$p_1 = p_2 = q_3 = p_6 = 0. \tag{1.86}$$

Equations (1.83, 86) now give

$$u_1 = \epsilon l, \ u_2 = 3\epsilon l, \ u_3 = 2\epsilon l, \ u_4 = 2\epsilon l - p_5, \ u_5 = 5\epsilon l + p_8, \ u_6 = 6\epsilon l, \tag{1.87}$$

and

$$\left.\begin{aligned}
&p_1 = 0, \ q_1 = 2\epsilon l; \quad p_2 = 0, \ q_2 = 4\epsilon l; \quad p_3 = 4\epsilon l, \ q_3 = 0; \quad p_4 = 2\epsilon l, \ q_4 = 0;\\
&q_5 = 2\epsilon l - p_5; \quad p_6 = 0, \ q_6 = 4\epsilon l; \quad p_7 = 3\epsilon l + p_5 + p_8, \ q_7 = \epsilon l - p_5 - p_8;\\
&q_8 = 2\epsilon l - p_8,
\end{aligned}\right\} \tag{1.88}$$

which by (1.82) imply

$$p_5 \geqslant 0, \quad p_8 \geqslant 0, \quad p_5 + p_8 \leqslant \epsilon l. \tag{1.89}$$

This solution is indeterminate. Basic solutions can be obtained by writing (i) $p_5 = p_8 = 0$, (ii) $p_5 = 0, p_8 = \epsilon l$ and hence $q_7 = 0$, or (iii) $p_5 = \epsilon l, p_8 = 0$ and hence $q_7 = 0$. The resulting solutions are degenerate and give the following layouts:

(i) members 1, 2, 5, 6, and 8 in tension, members 3 and 4 in compression,
(ii) members 1, 2, 5, and 6 in tension, members 3, 4, and 7 in compression, and
(iii) members 1, 2, 6, and 8 in tension, members 3, 4, and 7 in compression.

Non-degenerate basic solutions can be obtained by writing (iv) $p_5 = 0, 0 < p_8 < \epsilon l$, (v) $p_8 = 0, 0 < p_5 < \epsilon l$ and (vi) $p_5 + p_8 = \epsilon l$, $0 < p_5 < \epsilon l$ and hence $q_7 = 0$. The corresponding layouts are:

(iv) members 1, 2, 5, and 6 in tension, members 3 and 4 in compression,
(v) members 1, 2, 6, and 8 in tension, members 3 and 4 in compression, and
(vi) members 1, 2, and 6 in tension, members 3, 4, and 7 in compression.

The final type of solution can be obtained by writing (vii) $p_5 > 0, p_8 > 0$, $p_5 + p_8 < \epsilon l$. This gives the layout:

(vii) members 1, 2, and 6 in tension, members 3 and 4 in compression.

Consideration of the equilibrium equations corresponding to u_4 and u_5 removes members 5, 7, and 8, together with their common node, from the

layouts (i) to (vi) and reduces them all to layout (vii). The principle of virtual work gives for the remaining members, using (1.79),

$$T_1u_1 + T_2u_3/\sqrt{2} + T_3(u_1-u_2)/\sqrt{2} + T_4(u_3-u_2) + T_6(u_6-u_1-u_2)/\sqrt{2} = F(u_3+u_6)/2, \tag{1.90}$$

and so the remaining equilibrium equations are

$$\left.\begin{aligned} T_1 + T_3/\sqrt{2} - T_6/\sqrt{2} &= 0,\\ -T_3/\sqrt{2} - T_4 - T_6/\sqrt{2} &= 0,\\ T_2/\sqrt{2} + T_4 &= F/2,\\ T_6/\sqrt{2} &= F/2, \end{aligned}\right\} \tag{1.91}$$

which can be solved to give

$$T_1 = F + T_4,\quad T_2 = F/\sqrt{2} - \sqrt{2}T_4,\quad T_3 = -F/\sqrt{2} - \sqrt{2}T_4,\quad T_6 = F/\sqrt{2}. \tag{1.92}$$

Finally, the specification of layout (vii) implies

$$-F/2 \leqslant T_4 \leqslant 0. \tag{1.93}$$

The optimum framework is thus redundant, but the value of the redundant load T_4 must satisfy (1.93), so that the Michell conditions of (1.70) are satisfied. The extremes of (1.93) give $T_4 = 0$ and $T_4 = -F/2$ or $T_3 = 0$. The simplest layouts thus have members 1, 2, and 6 in tension and either member 3 or member 4 in compression. The combined layout gives an optimum structure, so long as both members 3 and 4 carry compressive loads. The volume of material for all these optimum structures is given by (1.85). All the designs are both optimum elastic and optimum plastic frameworks.

This example is interesting in that it shows that any basic solution (i) to (vi), which might be found by a purely numerical application of the simplex method,† gives the redundant optimum structure. This would not be obtained by a similar analysis of (1.8–10).

1.8 Approximation to absolute optima

The methods of analysis considered in this chapter are confined by their nature to the selection of optimum frameworks from a framework of given layout. A search for absolute optima requires consideration of all possible layouts and hence a selection from an infinite set of possible frameworks. An approximation to this ideal can be achieved by covering the region of space in which the required structure is allowed to lie by, for example, a closely spaced rectangular grid of nodal points and allowing members to lie along all segments joining nodal

† For example by the use of a standard programme on the digital computer.

points. The nodal points must include the points of application of given forces or of discrete approximations to given continuously distributed forces and all points of support.

Let (x_i, y_i, z_i) be the coordinates of the nodal points, l_{ij} the length of the segment (i, j) given by

$$l_{ij} = \{(x_j-x_i)^2 + (y_j-y_i)^2 + (z_j-z_i)^2\}^{1/2}, \tag{1.94}$$

A_{ij} the cross-sectional area of the member (i, j), T_{ij} the end load carried by the member (i, j), (X_i, Y_i, Z_i) the external force at node i, $\sigma = \sigma_T = \sigma_C$ the allowable stress and V the volume of material. The primal problem can be formulated as

$$\min V = \sum_{\substack{\text{pairs } i,\, j\\ i\neq j}} l_{ij}A_{ij}, \tag{1.95}$$

subject to

$$\sum_{\substack{j\\ j\neq i}} T_{ij}(x_j-x_i,\ y_j-y_i,\ z_j-z_i)/l_{ij} + (X_i, Y_i, Z_i) = 0 \quad (\text{all } i), \tag{1.96}$$

and

$$|T_{ij}| \leqslant \sigma A_{ij} \quad (\text{all pairs } i, j). \tag{1.97}$$

This formulation can now be converted to that of (1.1–1.4) by an appropriate change of notation and the omission of equations involving forces of constraint.

Let (u_i, v_i, w_i) be the virtual displacement at node i, ϵ the limiting strain and W the virtual work divided by $\sigma\epsilon$. The dual problem can be formulated as:

$$\max W = \sum_i (X_iu_i + Y_iv_i + Z_iw_i)/\sigma\epsilon, \tag{1.98}$$

subject to

$$|(u_j-u_i)(x_j-x_i) + (v_j-v_i)(y_j-y_i) + (w_j-w_i)(z_j-z_i)| \leqslant \epsilon l_{ij}^2 \quad (\text{all } i, j). \tag{1.99}$$

This formulation can now be converted to that of (1.65, 66) by an appropriate change of notation.

Calculations of the size envisaged here can only be carried out using digital computers. Many standard programmes exist for the solution of the linear programming problems which arise here. The compilation by Kienzi *et al.* (1968) has already been mentioned. A programme which uses this dual formulation and which is specially adapted to the solution of framework optimization problems is given in Hemp and Chan (1966). Results obtained using this programme are given below.

The problem of designing an optimum structure to transmit a force F to two fixed supports is illustrated in Fig. 1.3.† The structure is assumed to be symmetrical about the horizontal line through the point of application of F.

† Taken from Hemp and Chan (1966).

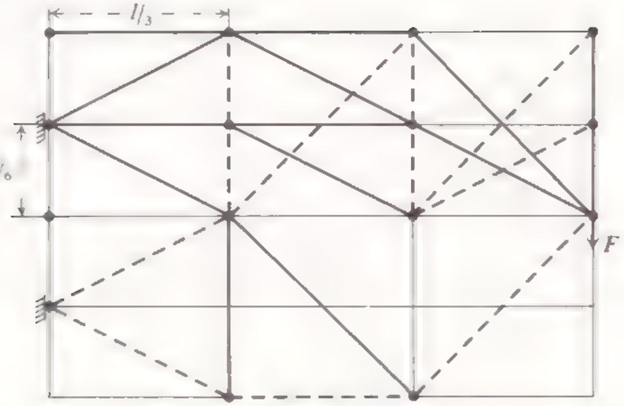


Fig. 1.3

The assumed grid of nodal points is shown and these are taken to have a completely general virtual motion consistent with symmetry. The solution to the dual problem of (1.98, 99) gives strains $\pm\epsilon$ in the members shown in the upper-half of the diagram. The full lines have strain $+\epsilon$ and are potential tension members, the dotted lines have strain $-\epsilon$ and are potential compression members. Several of the members are seen to be unnecessary to carry the load F, but those that remain make up a structure with order of redundancy three. A statically determinate optimum structure is shown in the lower half of Fig. 1.3. The volume of material in all these optimum structures is given by

$$V_{\min} = 4{\cdot}8\dot{3}\,Fl/\sigma. \tag{1.100}$$

The same problem as that of Fig. 1.3 is illustrated in Fig. 1.4,† but here a finer mesh of nodal points is used. The upper half of the diagram shows all the

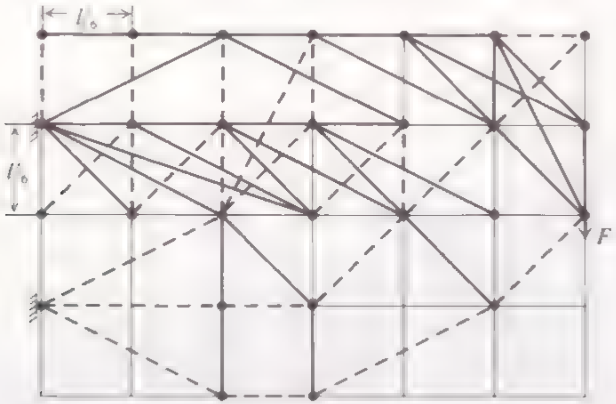


Fig. 1.4

† Taken from Hemp (1964).

potential members and the lower half a statically determinate optimum structure. The volume of material required at the optimum is:

$$V_{\min} = 4{\cdot}6\dot{1}\,Fl/\sigma. \tag{1.101}$$

It is interesting to compare (1.100, 101) with the exact solution obtained in section 4.9 below, which gives

$$V_{\min} = 4{\cdot}34\,..\,Fl/\sigma. \tag{1.102}$$

The best approximation is thus some six per cent above this exact value.

Another example of a computer designed structure is shown in Fig. 1.5.†

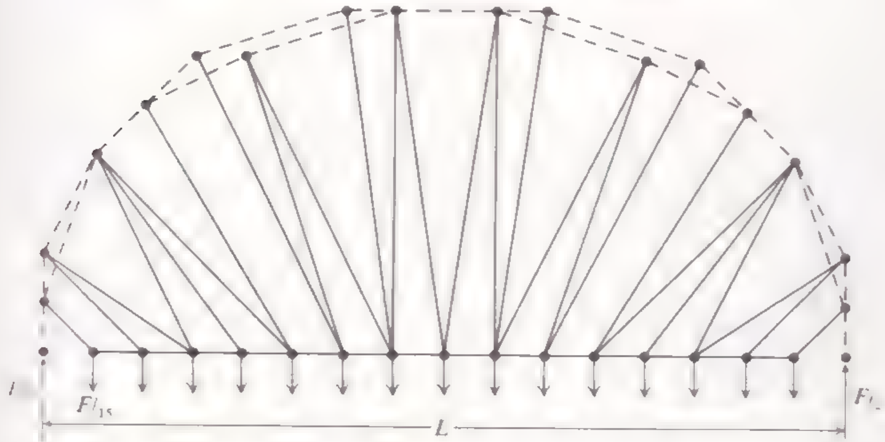


Fig. 1.5

Here the problem is to design an optimum structure to transmit a load F, uniformly distributed over a span L, to two vertical supports at the end of the span. The distributed load is represented by fifteen concentrated loads as shown. The resulting structure of arch, hangers and tie has a volume given by

$$V_{\min} = 1{\cdot}067\,..\,FL/\sigma. \tag{1.103}$$

1.9 Alternative loading conditions

Practical structures are usually designed to withstand more than one system of given forces. The present theory must be extended to deal with this more complex problem.

A single system of given forces has been denoted by $[F_j]$. Several systems can be written $[F_{\alpha j}]$, where the subscript α picks out a particular system by taking values $1, 2, \ldots s$. New quantities $[T_{\alpha i}]$, $[T'_{\alpha i}]$, $[T''_{\alpha i}]$ can be introduced in the same way to denote the corresponding end loads in members and slack

† This was obtained by Dr. W. J. Supple while working at Oxford under contract to the Ministry of Technology (1968–70).

variables. Generalizations of (1.1) and (1.6) will be valid for the new variables, with of course $[A_i]$ unchanged, since there is only one structure to be designed. It is convenient in the present case to retain the variables $[A_i]$, rather than to eliminate them as in (1.8–10), and to express the primal problem in the form:

$$\min V = \sum_{i=1}^{m} A_i l_i, \tag{1.104}$$

subject to

$$\sum_{i=1}^{m} K_{ij}(\sigma_T T'_{\alpha i} - \sigma_C T''_{\alpha i})/\sigma = F_{\alpha j} \qquad (j = 1, 2 \,..\, n;\ \alpha = 1, 2 \,..\, s), \tag{1.105}$$

$$T'_{\alpha i} + T''_{\alpha i} = \sigma A_i \qquad (i = 1, 2 \,..\, m;\ \alpha = 1, 2 \,..\, s), \tag{1.106}$$

and

$$T'_{\alpha i},\ T''_{\alpha i},\ A_i \geqslant 0 \qquad (i = 1, 2 \,..\, m;\ \alpha = 1, 2 \,..\, s). \tag{1.107}$$

This is a standard problem of linear programming of the same form as (1.11–13) and can be solved using the simplex method. It leads to larger calculations than (1.8–10) for a single system of forces.

The dual can be formulated using (1.50, 51), but as an illustration of the alternative Lagrangian approach of (1.62), the Lagrangian V^* corresponding to (1.104–106) will be written down and the dual deduced directly from it. Introducing Lagrangian multipliers $[u_{\alpha j}]/\sigma\epsilon$ and $[l_i \gamma_{\alpha i}]/\sigma\epsilon$ where $[u_{\alpha j}]$ are virtual displacements then gives

$$\begin{aligned} V^* = \sum_{i=1}^{m} A_i l_i + \sum_{\alpha=1}^{s} \sum_{j=1}^{n} (u_{\alpha j}/\sigma\epsilon) \left\{ F_{\alpha j} - \sum_{i=1}^{m} K_{ij}(\sigma_T T'_{\alpha i} - \sigma_C T''_{\alpha i})/\sigma \right\} + \\ + \sum_{\alpha=1}^{s} \sum_{i=1}^{m} (l_i \gamma_{\alpha i}/\sigma\epsilon) \left\{ T'_{\alpha i} + T''_{\alpha i} - \sigma A_i \right\}. \end{aligned} \tag{1.108}$$

The conditions that V^* should have a minimum for the variables $[A_i]$, $[T'_{\alpha i}]$ and $[T''_{\alpha i}]$ are then

$$\left(\epsilon - \sum_{\alpha=1}^{s} \gamma_{\alpha i} \right) A_i = 0, \quad \epsilon \geqslant \sum_{\alpha=1}^{s} \gamma_{\alpha i} \qquad (i = 1, 2 \,..\, m), \tag{1.109}$$

$$(\gamma_{\alpha i} - \sigma_T \epsilon_{\alpha i}/\sigma) T'_{\alpha i} = 0, \quad \gamma_{\alpha i} \geqslant \sigma_T \epsilon_{\alpha i}/\sigma \qquad (i = 1, 2 \,..\, m;\ \alpha = 1, 2 \,..\, s), \tag{1.110}$$

and

$$(\gamma_{\alpha i} + \sigma_C \epsilon_{\alpha i}/\sigma) T''_{\alpha i} = 0, \quad \gamma_{\alpha i} \geqslant -\sigma_C \epsilon_{\alpha i}/\sigma \qquad (i = 1, 2 \,..\, m;\ \alpha = 1, 2 \,..\, s), \tag{1.111}$$

where $\{\epsilon_{\alpha i}\}$ are member virtual strains given by

$$\epsilon_{\alpha i} = \sum_{j=1}^{n} K_{ij} u_{\alpha j}/l_i \qquad (i = 1, 2 \,..\, m;\ \alpha = 1, 2 \,..\, s). \tag{1.112}$$

The equations in (1.109–111) are another way of writing relations of type (1.63), while the inequations express conditions, which prevent $V^* \to -\infty$, as one of the non-negative variables A_i, $T'_{\alpha i}$, $T''_{\alpha i}$ tends to $+\infty$. The dual problem, which is a maximum problem, is obtained by imposing (1.109–111) on V^* to give the objective function W, which is seen to depend upon the variables $[u_{\alpha j}]$ alone, and by eliminating $[A_i]$, $[T'_{\alpha i}]$ and $[T''_{\alpha i}]$ from (1.109–111) to give the dual constraints. The dual problem can thus be written

$$\max W = \sum_{\alpha=1}^{s} \sum_{j=1}^{n} F_{\alpha j} u_{\alpha j}/\sigma\epsilon, \tag{1.113}$$

subject to

$$\epsilon \geqslant \sum_{\alpha=1}^{s} \gamma_{\alpha i}, \quad \gamma_{\alpha i} \geqslant \sigma_T \epsilon_{\alpha i}/\sigma, \quad \gamma_{\alpha i} \geqslant -\sigma_C \epsilon_{\alpha i}/\sigma \qquad (i = 1, 2 \,..\, m;\ \alpha = 1, 2 \,..\, s). \tag{1.114}$$

The function W of (1.113) is the sum of the virtual work of the various systems of given forces taken over their corresponding displacements. The quantities $[\gamma_{\alpha i}]$ can be eliminated from (1.114) to give

$$\sum_{\alpha=1}^{s} (\sigma_T \epsilon_{\alpha i}/\sigma \quad \text{or} \quad -\sigma_C \epsilon_{\alpha i}/\sigma) \leqslant \epsilon \qquad (i = 1, 2 \,..\, m), \tag{1.115}$$

where all the 2^s alternative relations are implied. All these can be obtained from (1.114) by adding an appropriate selection of the second and third relations of (1.114) to the first. Conversely (1.114) can be obtained from (1.115) by writing†

$$\gamma_{\alpha i} = \max(\sigma_T \epsilon_{\alpha i}/\sigma, \ -\sigma_C \epsilon_{\alpha i}/\sigma) \qquad (i = 1, 2 \,..\, m;\ \alpha = 1, 2 \,..\, s), \tag{1.116}$$

since, for each i, one of the 2^s relations (1.115) will give the first of (1.114) and (1.116) satisfies the second and third of (1.114). The relations (1.115) can thus be taken as the constraints for the dual. They are $2^s m$ in number and show how rapidly the size of the present problem grows with the number s of alternative systems of given forces.

The relation (1.115) can also be written

$$\sum_{\substack{\alpha \\ \epsilon_{\alpha i} > 0}} \sigma_T |\epsilon_{\alpha i}|/\sigma + \sum_{\substack{\alpha \\ \epsilon_{\alpha i} < 0}} \sigma_C |\epsilon_{\alpha i}|/\sigma \leqslant \epsilon \qquad (i = 1, 2 \,..\, m), \tag{1.117}$$

or, if $\sigma_T = \sigma_C = \sigma$,

$$\sum_{\alpha=1}^{s} |\epsilon_{\alpha i}| \leqslant \epsilon \qquad (i = 1, 2 \,..\, m), \tag{1.118}$$

which shows the constraints imposed on the virtual strains in a particularly clear form.

† The choice of $\gamma_{\alpha i}$, for given i, is unique if (1.115) is satisfied as an equality; otherwise it is indeterminate.

The relations between the optimum solutions of the primal and the dual enable a number of deductions to be made about the optimum structure. The following cases can arise:

(i) $T'_{\alpha i} > 0$, $T''_{\alpha i} > 0$, which by (1.106) implies $A_i > 0$ and gives a stress between σ_T and $-\sigma_C$.† Also since the corresponding dual constraints must be equalities (1.114) gives $\gamma_{\alpha i} = \sigma_T \epsilon_{\alpha i}/\sigma = -\sigma_C \epsilon_{\alpha i}/\sigma$ and so $\epsilon_{\alpha i} = \gamma_{\alpha i} = 0$.

(ii) $T'_{\alpha i} > 0$, $T''_{\alpha i} = 0$, which implies $A_i > 0$ and gives a stress σ_T.† Also $\gamma_{\alpha i} = \sigma_T \epsilon_{\alpha i}/\sigma \geqslant -\sigma_C \epsilon_{\alpha i}/\sigma$ and so $\epsilon_{\alpha i} \geqslant 0$ and $\gamma_{\alpha i} \geqslant 0$.

(iii) $T'_{\alpha i} = 0$, $T''_{\alpha i} > 0$, which implies $A_i > 0$ and gives a stress $-\sigma_C$.† Also $\gamma_{\alpha i} = -\sigma_C \epsilon_{\alpha i}/\sigma \geqslant \sigma_T \epsilon_{\alpha i}/\sigma$ and so $\epsilon_{\alpha i} \leqslant 0$ and $\gamma_{\alpha i} \geqslant 0$.

(iv) $T'_{\alpha i} = T''_{\alpha i} = 0$, which implies $A_i = 0$. Also $\gamma_{\alpha i} \geqslant 0$ and $-\sigma\gamma_{\alpha i}/\sigma_C \leqslant \epsilon_{\alpha i} \leqslant \sigma\gamma_{\alpha i}/\sigma_T$.

The case $A_i > 0$ gives $\sum_{\alpha=1}^{s} \gamma_{\alpha i} = \epsilon$ and so it follows that every member that is present must be stressed to the limit by at least one system of given forces. A member that is stressed to the limit in tension will have a corresponding positive virtual strain, a member that is stressed to the limit in compression will have a corresponding negative virtual strain and a member that is not stressed to the limit will have a corresponding zero virtual strain. A member that is absent can have a strain of any sign. A member that is present will give an equality in (1.117) and so its virtual strains must satisfy

$$\sum_{\substack{\alpha \\ \text{Stress} = \sigma_T}} \sigma_T|\epsilon_{\alpha i}|/\sigma + \sum_{\substack{\alpha \\ \text{Stress} = -\sigma_C}} \sigma_C|\epsilon_{\alpha i}|/\sigma = \epsilon \qquad (i = 1, 2 \ldots m;\ A_i > 0). \tag{1.119}$$

For a member with $A_i = 0$, the relation (1.117) must be satisfied by its virtual strains.

The generalization of the Michell type theorem of (1.70) is as follows:

The necessary and sufficient conditions that a pin-jointed framework, selected from a given framework, should have a minimum volume of material are (i) that it should be capable of safely carrying its alternative systems of forces, (ii) that each member should be stressed to the limit σ_T or $-\sigma_C$ by at least one system of force, (iii) that, corresponding to each system of force, the framework allows a virtual displacement of its nodes, which produces non-negative strains in members loaded to stress σ_T, non-positive strains in members loaded to a stress $-\sigma_C$ and zero strains in members not stressed to these limits, (iv) that the strains of (iii) satisfy equation (1.119) and (v) that the strains in all members (including those not present in the optimum structure), corresponding to the displacements of (iii), satisfy the relation (1.117). (1.120)

The general theory developed above is complete, but clearly likely to lead to quite large calculations, compared with those for design to a single system of

† This follows from $T_{\alpha i}/A_i = (\sigma_T T'_{\alpha i} - \sigma_C T''_{\alpha i})/(T'_{\alpha i} + T''_{\alpha i})$.

given forces. In the special case of two systems of given forces ($s = 2$), with $\sigma_T = \sigma_C = \sigma$, considerable simplification is possible. This is due to the fact that the problems can, in this special case, be reduced to two problems of optimum design with a single given system of forces for each.

The relations (1.115) with $s = 2$ and $\sigma_T = \sigma_C = \sigma$ give

$$\epsilon_{1i} + \epsilon_{2i} \leqslant \epsilon,\ \epsilon_{1i} - \epsilon_{2i} \leqslant \epsilon,\ -\epsilon_{1i} + \epsilon_{2i} \leqslant \epsilon,\ -\epsilon_{1i} - \epsilon_{2i} \leqslant \epsilon \qquad (i = 1, 2 \ldots m), \tag{1.121}$$

which suggest the change of notation

$$\left.\begin{aligned} U_{1j} &= u_{1j} + u_{2j},\ U_{2j} = u_{1j} - u_{2j} \qquad (j = 1, 2 \ldots n), \\ E_{1i} &= \epsilon_{1i} + \epsilon_{2i},\ E_{2i} = \epsilon_{1i} - \epsilon_{2i} \qquad (i = 1, 2 \ldots m). \end{aligned}\right\} \tag{1.122}$$

Equations (1.112, 122) give

$$E_{\alpha i} = \sum_{j=1}^{n} K_{ij} U_{\alpha j}/l_i \qquad (i = 1, 2 \ldots m;\ \alpha = 1, 2), \tag{1.123}$$

while (1.121) can be written

$$E_{\alpha i} \leqslant \epsilon, \quad -E_{\alpha i} \leqslant \epsilon \qquad (i = 1, 2 \ldots m;\ \alpha = 1, 2), \tag{1.124}$$

and (1.113) as

$$\max W = \sum_{\alpha=1}^{2} W_\alpha, \tag{1.125}$$

where

$$W_\alpha = \sum_{j=1}^{n} f_{\alpha j} U_{\alpha j}/\sigma\epsilon \qquad (\alpha = 1, 2), \tag{1.126}$$

and

$$f_{1j} = (F_{1j} + F_{2j})/2, \quad f_{2j} = (F_{1j} - F_{2j})/2 \qquad (j = 1, 2 \ldots n). \tag{1.127}$$

The present problem thus splits into two, namely to maximize W_α ($\alpha = 1, 2$) of (1.126) subject to the constraints of (1.123, 124). The given forces for these problems are, by (1.127), equal to half the sum and to half the difference of the two systems of given forces.

The theory of the present section, like that of section 1.1, has been developed on the basis of plastic design. The conditions for optimum structures that have been found do not yield optimum elastic designs, except in trivial cases, such as when each system of given forces is carried by separate non-interacting frameworks.

1.10 An example with two alternative loading conditions

Consider the given layout of Fig. 1.2 and let two given systems of force be specified by

$$\left.\begin{aligned} F_{1j} &= 0 \quad (j = 1, 2 \ldots 5), \qquad F_{16} = 2F, \\ F_{2j} &= 0 \quad (j = 1, 2, 4, 5, 6), \quad F_{23} = 2F. \end{aligned}\right\} \tag{1.128}$$

Equations (1.127, 128) give

$$\left.\begin{aligned} f_{1j} &= 0 \quad (j = 1, 2, 4, 5), \; f_{13} = f_{16} = F, \\ f_{2j} &= 0 \quad (j = 1, 2, 4, 5), \; f_{23} = -F, \; f_{26} = F. \end{aligned}\right\} \tag{1.129}$$

The system $[f_{1j}]$ defines the problem solved in section 1.7 and so by (1.85, 87, 89), with an appropriate change of notation, a basic solution to this first problem can be written:

$$\left.\begin{aligned} (W_1)_{\max} &= 8Fl/\sigma, \\ U_{1j} &= \epsilon l\,(1, 3, 2, 2, 5, 6). \end{aligned}\right\} \tag{1.130}$$

A solution to the second problem with forces $[f_{2j}]$ can be obtained by solving (1.81) in the form

$$\left.\begin{aligned} &u_1 = \epsilon l - p_1, \; q_1 = 2\epsilon l - p_1, \\ &u_3 = -2\epsilon l + q_2, \; p_2 = 4\epsilon l - q_2, \\ &u_2 = -\epsilon l + q_2 - q_4, \; p_4 = 2\epsilon l - q_4, \\ &p_3 = p_1 + q_2 - q_4, \; q_3 = 4\epsilon l - p_1 - q_2 + q_4, \\ &u_4 = 2\epsilon l - p_1 - p_5, \; q_5 = 2\epsilon l - p_5, \\ &u_6 = 2\epsilon l - p_1 + q_2 - q_4 - p_6, \; q_6 = 4\epsilon l - p_6, \\ &u_5 = \epsilon l - p_1 + q_2 - q_4 - p_6 + p_8, \; q_8 = 2\epsilon l - p_8, \\ &p_7 = 3\epsilon l - q_4 + p_5 - p_6 + p_8, \; q_7 = \epsilon l + q_4 - p_5 + p_6 - p_8, \end{aligned}\right\} \tag{1.131}$$

which gives for W the expression

$$W = F(u_6 - u_3)/\sigma\epsilon = F(4\epsilon l - p_1 - q_4 - p_6)/\sigma\epsilon. \tag{1.132}$$

The maximum is $4Fl/\sigma$ with $p_1 = q_4 = p_6 = 0$ and so a basic solution to the second problem can be written, on changing to the notation of (1.130) and using (1.131), in the form:

$$\left.\begin{aligned} (W_2)_{\max} &= 4Fl/\sigma, \\ U_{2j} &= \epsilon l\,(1, -1, -2, 2, 1, 2). \end{aligned}\right\} \tag{1.133}$$

The total volume of material required for the present problem of alternative loading is by (1.125, 130, 133)

$$W_{\max} = 12Fl/\sigma. \tag{1.134}$$

The virtual displacements follow from (1.122, 130, 133) as

$$\left.\begin{aligned} u_{1j} &= \epsilon l\,(1, 1, 0, 2, 3, 4), \\ u_{2j} &= \epsilon l\,(0, 2, 2, 0, 2, 2), \end{aligned}\right\} \tag{1.135}$$

and so by (1.79) the virtual strains can be written

$$\left.\begin{aligned} \epsilon_{1i} &= \epsilon\,(1, 0, 0, -1, 1, 1, -1/2, 1), \\ \epsilon_{2i} &= \epsilon\,(0, 1, -1, 0, 0, 0, 0, 0). \end{aligned}\right\} \tag{1.136}$$

Inspection of (1.136) shows that (1.118) is satisfied as an equality for all i, except $i = 7$. This means that $A_7 = 0$ and hence that $T_{17} = T_{27} = 0$. Equilibrium then shows that $T_{15} = T_{18} = 0$ and $T_{25} = T_{28} = 0$. It thus follows that $A_5 = A_8 = 0$ and so the three members $i = 5, 7, 8$ can be omitted.

Equation (1.116) shows that here $\gamma_{\alpha i} = |\epsilon_{\alpha i}|$ $(\alpha = 1, 2; i \neq 7)$ and so $\gamma_{\alpha i} - \epsilon_{\alpha i} > 0$, if $\epsilon_{\alpha i} < 0$ and $\gamma_{\alpha i} + \epsilon_{\alpha i} > 0$ if $\epsilon_{\alpha i} > 0$. In the first case the second of (1.114) is satisfied as a strict inequality and the corresponding primal variable $T'_{\alpha i} = 0$. In the second case the same is true of the third of (1.114) and so $T''_{\alpha i} = 0$. Application of these results to (1.136) gives

$$\left.\begin{aligned} T''_{11} &= T'_{14} = T''_{16} = 0, \\ T''_{22} &= T'_{23} = 0. \end{aligned}\right\} \tag{1.137}$$

The equilibrium equations (1.91) adapted to the present cases give, writing $T_i = T_i' - T_i''$, using (1.137), and changing to the present notation

$$\left.\begin{aligned} T'_{11} + (T'_{13} - T''_{13})/\sqrt{2} - T'_{16}/\sqrt{2} &= 0, \\ -(T'_{13} - T''_{13})/\sqrt{2} + T''_{14} - T'_{16}/\sqrt{2} &= 0, \\ (T'_{12} - T''_{12})/\sqrt{2} - T''_{14} &= 0, \\ T'_{16}/\sqrt{2} &= F, \end{aligned}\right\} \tag{1.138}$$

and

$$\left.\begin{aligned} T'_{21} - T''_{21} - T''_{23}/\sqrt{2} - (T'_{26} - T''_{26})/\sqrt{2} &= 0, \\ T''_{23}/\sqrt{2} - (T'_{24} - T''_{24}) - (T'_{26} - T''_{26})/\sqrt{2} &= 0, \\ T'_{22}/\sqrt{2} + (T'_{24} - T''_{24}) &= F, \\ (T'_{26} - T''_{26})/\sqrt{2} &= 0. \end{aligned}\right\} \tag{1.139}$$

Equations (1.106) give, using (1.137),

$$\left.\begin{aligned} \sigma A_1 &= T'_{11} = T'_{21} + T''_{21}, \\ \sigma A_2 &= T'_{12} + T''_{12} = T'_{22}, \\ \sigma A_3 &= T'_{13} + T''_{13} = T''_{23}, \\ \sigma A_4 &= T''_{14} = T'_{24} + T''_{24}, \\ \sigma A_6 &= T'_{16} = T'_{26} + T''_{26}. \end{aligned}\right\} \tag{1.140}$$

Equations (1.137–140), together with the non-negativity conditions (1.107), can be solved to give

$$\left.\begin{aligned} &T'_{11} = 2F - T''_{14}, && T''_{11} = 0, \\ &T'_{12} = \sqrt{2}T''_{14}, && T''_{12} = 0, \\ &T'_{13} = 0, && T''_{13} = \sqrt{2}(F - T''_{14}), \\ &T'_{14} = 0, && F/2 \leqslant T''_{14} \leqslant F, \\ &T'_{16} = \sqrt{2}F, && T''_{16} = 0, \end{aligned}\right\} \qquad (1.141)$$

$$\left.\begin{aligned} &T'_{21} = 3F/2 - T''_{14}, && T''_{21} = F/2, \\ &T'_{22} = \sqrt{2}T''_{14}, && T''_{22} = 0, \\ &T'_{23} = 0, && T''_{23} = \sqrt{2}(F - T''_{14}), \\ &T'_{24} = F/2, && T''_{24} = T''_{14} - F/2, \\ &T'_{26} = F/\sqrt{2}, && T''_{26} = F/\sqrt{2}, \end{aligned}\right\} \qquad (1.142)$$

and

$$\left.\begin{aligned} A_1 = (2F - T''_{14})/\sigma, \quad A_2 = \sqrt{2}T''_{14}/\sigma, \quad A_3 = \sqrt{2}(F - T''_{14})/\sigma, \\ A_4 = T''_{14}/\sigma, \quad A_6 = \sqrt{2}F/\sigma. \end{aligned}\right\} \qquad (1.143)$$

The end loads in the members for the two loading cases follow from (1.141, 142) as

$$\left.\begin{aligned} &T_{11} = 2F - T''_{14}, \quad T_{12} = \sqrt{2}T''_{14}, \quad T_{13} = -\sqrt{2}(F - T''_{14}), \quad T_{14} = -T''_{14}, \quad T_{16} = \sqrt{2}F, \\ &T_{21} = F - T''_{14}, \quad T_{22} = \sqrt{2}T''_{14}, \quad T_{23} = -\sqrt{2}(F - T''_{14}), \quad T_{24} = F - T''_{14}, \quad T_{26} = 0. \end{aligned}\right\} \qquad (1.144)$$

where, by (1.141),

$$F/2 \leqslant T''_{14} \leqslant F. \qquad (1.145)$$

The optimum solution obtained above is a redundant structure, except when $T''_{14} = F$ and $T_{13} = T_{23} = A_3 = 0$. It gives a range of designs depending on the parameter T''_{14}. In the first system of loading all members are stressed to the limits $\pm\sigma$. In the second system members 2 and 3 are stressed to the limits, while members 1, 4 and 6 are understressed. It may be verified that (1.143) gives a V equal to W_{max} of (1.134) and that all the conditions of (1.120) are satisfied.

1.11 Constrained minima of functions of non-negative variables

Consider the problem

$$\min z = f(x_1, x_2 \ldots x_n;\ y_1, y_2 \ldots y_m), \qquad (1.146)$$

subject to

$$\phi_j(x_1, x_2 \ldots x_n;\ y_1, y_2 \ldots y_m) = 0 \qquad (j = 1, 2 \ldots p;\ p < m + n), \qquad (1.147)$$

and

$$y_k \geqslant 0 \qquad (k = 1, 2 \ldots m). \qquad (1.148)$$

Assume that values $[x_i]$, $[y_k]$ are known that make z of (1.146) a minimum.† It then follows, for all small variations $[\delta x_i]$ and $[\delta y_k]$, which by (1.147, 148) must satisfy

$$\sum_{i=1}^{n} \phi_{jx_i}\, \delta x_i + \sum_{k=1}^{m} \phi_{jy_k}\, \delta y_k = 0 \qquad (j = 1, 2 \ldots p), \qquad (1.149)$$

and

$$\delta y_k \geqslant 0 \qquad (\text{all } k \text{ such that } y_k = 0), \qquad (1.150)$$

that

$$\sum_{i=1}^{n} f_{x_i}\, \delta x_i + \sum_{k=1}^{m} f_{y_k}\, \delta y_k \geqslant 0. \qquad (1.151)$$

Only first order terms have been written in (1.149–151), since the arguments to be developed depend upon the signs of certain quantities, which are not affected, for small enough variations, by higher order terms.

Introducing Lagrangian multipliers $[\lambda_j]$ leads, using (1.149, 151) to

$$\sum_{i=1}^{n} \left(f_{x_i} + \sum_{j=1}^{p} \lambda_j \phi_{jx_i}\right) \delta x_i + \sum_{k=1}^{m} \left(f_{y_k} + \sum_{j=1}^{p} \lambda_j \phi_{jy_k}\right) \delta y_k \geqslant 0, \qquad (1.152)$$

which is valid for all $[\delta x_i]$, $[\delta y_k]$, which satisfy (1.149, 150).

Assume now that (1.149) can be solved for $\delta x_i (i = i_1, i_2 \ldots i_s)$, $\delta y_k (k = k_1, k_2 \ldots k_t)$, where $s + t = p$ and the selection from $[\delta y_k]$ is such that

$$y_k > 0 \qquad (k = k_1, k_2 \ldots k_t). \qquad (1.153)$$

This means that the matrix

$$[\phi_{jx_i}, \phi_{jy_k}] \text{ has rank } p \qquad (j = 1, 2 \ldots p;\ i = 1, 2 \ldots n;\ \text{all } k \text{ such that } y_k > 0), \qquad (1.154)$$

which is a 'regularity condition' for the present problem. The remaining variables in (1.149) are denoted by $\delta x_i (i = i_{s+1}, \ldots i_n)$, $\delta y_k (k = k_{t+1}, \ldots k_m)$ and these can be regarded as independent variables, taking any values, subject only to (1.150).

The multipliers are now chosen to satisfy the equations

$$\left.\begin{aligned} f_{x_i} + \sum_{j=1}^{p} \lambda_j \phi_{jx_i} = 0 \qquad (i = i_1, i_2 \ldots i_s), \\ f_{y_k} + \sum_{j=1}^{p} \lambda_j \phi_{jy_k} = 0 \qquad (k = k_1, k_2 \ldots k_t), \end{aligned}\right\} \qquad (1.155)$$

† The functions f and $[\phi_j]$ are assumed to be continuously differentiable in a neighbourhood of these values. A variable as a subscript is used to denote a derivative.

which is always possible, since the determinant of (1.155) is the same determinant as that assumed to be non-vanishing in the solution of (1.149). Substituting from (1.155) into (1.152) then gives

$$\sum_{i=i_{s+1}}^{i_n} \left(f_{x_i} + \sum_{j=1}^{p} \lambda_j \phi_{jx_i}\right)\delta x_i + \sum_{k=k_{t+1}}^{k_m} \left(f_{y_k} + \sum_{j=1}^{p} \lambda_j \phi_{jy_k}\right)\delta y_k \geqslant 0, \tag{1.156}$$

where the variations $\delta x_i (i = i_{s+1}, \ldots i_n)$, $\delta y_k (k = k_{t+1}, \ldots k_n)$ are independent variables.

Now (1.156) implies

$$\left.\begin{aligned} \left(f_{x_i} + \sum_{j=1}^{p} \lambda_j \phi_{jx_i}\right)\delta x_i \geqslant 0 \qquad & (i = i_{s+1}, \ldots i_n), \\ \left(f_{y_k} + \sum_{j=1}^{p} \lambda_j \phi_{jy_k}\right)\delta y_k \geqslant 0 \qquad & (k = k_{t+1}, \ldots k_m), \end{aligned}\right\} \tag{1.157}$$

since it is possible to set all but one of the variations in (1.156) equal to zero and leave the remaining one arbitrary. It then follows that

$$\left.\begin{aligned} f_{x_i} + \sum_{j=1}^{p} \lambda_j \phi_{jx_i} = 0 \qquad & (i = i_{s+1}, \ldots i_n), \\ f_{y_k} + \sum_{j=1}^{p} \lambda_j \phi_{jy_k} = 0 \qquad & (k = k_{t+1}, \ldots k_m \text{ and } k \text{ such that } y_k > 0), \end{aligned}\right\} \tag{1.158}$$

$$f_{y_k} + \sum_{j=1}^{p} \lambda_j \phi_{jy_k} \geqslant 0 \qquad (k = k_{t+1}, \ldots k_m \text{ and } k \text{ such that } y_k = 0), \tag{1.159}$$

since if one of the equations in (1.158) is not valid, then, by giving the corresponding δx_i or δy_k a value of appropriate sign in (1.157), a contradiction can be obtained and if (1.159) is false, then the corresponding inequation of (1.157) can be contradicted by giving the δy_k, which occurs in it, a positive value, which (1.150) allows.

A combination of (1.155, 158, 159) gives as necessary conditions for the minimum of z of (1.146), subject to (1.147, 148), the relations

$$f_{x_i} + \sum_{j=1}^{p} \lambda_j \phi_{jy_k} = 0 \qquad (i = 1, 2 \ldots n), \tag{1.160}$$

and

$$\left(f_{y_k} + \sum_{j=1}^{p} \lambda_j \phi_{jy_k}\right) y_k = 0, \quad f_{y_k} + \sum_{j=1}^{p} \lambda_j \phi_{jy_k} \geqslant 0 \qquad (k = 1, 2 \ldots m). \tag{1.161}$$

The relations (1.147, 160, 161) give $p + n + m$ equations for the $p + n + m$ variables $[\lambda_j]$, $[x_i]$, and $[y_k]$. The solution which gives minimum z, if it exists, is to be found among the solution to these equations and must also satisfy the inequations of (1.148, 161). It is not possible to say in advance which values of k give $y_k = 0$ and so the various alternatives have to be examined. It may be that (1.154) cannot be verified in advance and must be checked after a solution has been obtained. The process of solving (1.160, 161) for λ_j usually provides a proof of (1.154). Irregular or degenerate solutions for which (1.154) is not satisfied are possible. These can be avoided by making suitable small changes in the constraints of (1.147, 148).

The conditions of (1.160, 161) can also be expressed by the statement that they are the conditions that the Lagrangian

$$Z = f + \sum_{j=1}^{p} \lambda_j \phi_j \tag{1.162}$$

regarded as a function of $[x_i]$, $[y_k]$, with $[\lambda_j]$ as parameters, should have a minimum, subject to (1.148).

1.12 Structures of maximum stiffness

Consider, as in section 1.1, a framework of given layout, which is defined by the areas of cross-section $[A_i]$ of its members. Replace (1.2) by the condition that no A_i can be less than a positive limit A_0. This can be written

$$A_i = A_0 + p_i, \quad p_i \geqslant 0 \qquad (i = 1, 2 \ldots m). \tag{1.163}$$

The relation (1.163) is introduced so as to avoid degeneracy in the minimum problem to be considered. The limit $A_0 \to 0$ will be imposed at the end of the analysis.

Let the framework be loaded by forces $[F_j]$ and let the resulting displacements, corresponding to an elastic deformation, be denoted by $\{v_j\}$ to distinguish them from the virtual displacements $\{u_j\}$ of section 1.5, with which they are of course kinematically identical. Equation (1.68) gives for the strains $\{\epsilon_i\}$ in the members

$$\epsilon_i = \sum_{j=1}^{n} K_{ij} v_j / l_i \qquad (i = 1, 2 \ldots m). \tag{1.164}$$

The end loads $[T_i]$ follow from

$$T_i = EA_i \epsilon_i \qquad (i = 1, 2 \ldots m), \tag{1.165}$$

where E is Young's modulus for the material of construction. Equations (1.1, 164, 165) yield, on eliminating $\{\epsilon_i\}$ and $[T_i]$, the displacement form of the equilibrium equations

$$\sum_{i=1}^{m} \sum_{k=1}^{n} EA_i K_{ij} K_{ik} v_k / l_i = F_j \qquad (j = 1, 2 \ldots n). \tag{1.166}$$

Since all members of the original structural layout of Section 1.1 are present, it follows that equations (1.166) always yield a unique solution for $\{v_k\}$.

The problem of determining structures of maximum stiffness will be interpreted here as the problem of finding that structure of given volume of material V, which, under the action of the given forces, stores the least amount of strain energy U or in other words undergoes the least generalized displacement. Mathematically this can be stated as

$$\min U = \sum_{j=1}^{n} F_j v_j/2, \tag{1.167}$$

subject to

$$\sum_{i=1}^{m} A_i l_i = V, \tag{1.168}$$

and to relations (1.163, 166). The Lagrangian U^* for this problem can be written by (1.162) as

$$U^* = \sum_{k=1}^{n} F_k v_k/2 + \lambda\left(\sum_{i=1}^{m} A_i l_i - V\right) + \sum_{i=1}^{m} l_i \mu_i (A_0 + p_i - A_i) + \\ + \sum_{j=1}^{n} (\nu_j/2)\left(F_j - \sum_{i=1}^{m}\sum_{k=1}^{n} EA_i K_{ij} K_{ik} v_k/l_i\right), \tag{1.169}$$

where $\lambda, \{\mu_i\}, \{\nu_j\}$ are multipliers, and conditions for U^* to be a minimum, deduced by applying (1.160, 161) or directly, can be formulated as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} EA_i K_{ik} K_{ij} \nu_j / l_i = F_k \qquad (k = 1, 2 \ldots n), \tag{1.170}$$

$$\lambda = \sum_{j=1}^{n}\sum_{k=1}^{n} EK_{ij} K_{ik} \nu_j v_k / 2 l_i^2 + \mu_i \qquad (i = 1, 2 \ldots m), \tag{1.171}$$

and

$$\mu_i p_i = 0, \quad \mu_i \geqslant 0 \qquad (i = 1, 2 \ldots m). \tag{1.172}$$

Equations (1.170) for $\{\nu_j\}$ are identical with (1.166) for $\{v_k\}$ and since this last has a unique solution, it follows that (1.170) can be solved and that the solution is

$$\nu_j = v_j \qquad (j = 1, 2 \ldots n). \tag{1.173}$$

Equations (1.171) can now be solved for $\{\mu_i\}$ in terms of λ and an equation for λ can be found by writing $\mu_i = 0$ for any i for which $p_i > 0$ or, by (1.163), $A_i > A_0$. Since $A_0 \to 0$, there must be at least one such i, because otherwise

(1.168) would be contradicted. It thus follows that a set of equations for the present problem like (1.155) is soluble or that the regularity criterion of (1.154) is satisfied. The derivation of (1.170–172) is thus justified.

Substituting from (1.173) in (1.171) and then making use of (1.164), gives

$$\lambda = E\epsilon_i^2/2 + \mu_i \qquad (i = 1, 2 \ldots m). \tag{1.174}$$

Equation (1.172) shows that $\lambda \geqslant 0$ and so writing

$$\lambda = \bar{\sigma}^2/2E \tag{1.175}$$

and interpreting (1.172), using (1.163), it follows that

$$\left.\begin{array}{ll} \epsilon_i = \pm\bar{\sigma}/E & (A_i > A_0) \\ |\epsilon_i| \leqslant \bar{\sigma}/E & (A_i = A_0) \end{array}\right\} \quad (i = 1, 2 \ldots m), \tag{1.176}$$

or taking the limit $A_0 \to 0$,

$$\left.\begin{array}{ll} \epsilon_i = \pm\bar{\sigma}/E & (A_i > 0) \\ |\epsilon_i| \leqslant \bar{\sigma}/E & (A_i = 0) \end{array}\right\} \quad (i = 1, 2 \ldots m). \tag{1.177}$$

Equation (1.165) gives

$$T_i/A_i = \pm\bar{\sigma} \qquad (i = 1, 2 \ldots m;\ A_i > 0). \tag{1.178}$$

The structure of maximum stiffness thus satisfies the equilibrium equations (1.1), the conditions (1.177) for an allowable stress $\bar{\sigma}$ and also allows its own displacement $\{v_j\}$, which, by (1.177), gives a strain $\bar{\sigma}/E$ in its tension members, a strain $-\bar{\sigma}/E$ in its compression members and strains between these limits in members of the original layout, which are absent from this optimum structure. It follows by (1.70) that the present optimum structure of maximum stiffness is identical with the structure of least volume designed to carry the same loads $\{F_j\}$ with an allowable stress $\bar{\sigma}$.† The previous methods of resolving this last problem are thus available to solve the problem of maximum stiffness.

It only remains to determine $\bar{\sigma}$. This follows from (1.168). If the V_{min} for the equivalent structure of least volume is

$$V_{min} = \beta F l/\bar{\sigma}, \tag{1.179}$$

where F is a typical force, l a typical length and β a number, like the 7 of (1.41), determined by the optimum analysis, then

$$\bar{\sigma} = \beta F l / V, \tag{1.180}$$

where V is the given volume of (1.168).‡

† This result was first given by H. L. Cox, see Cox (1965)

‡ Safety requires $\bar{\sigma} < \sigma$, which can be satisfied by reducing F, if necessary, since this does not affect stiffness. If F is actually required to be carried by the structure, V must be not less than $\beta F l/\sigma$.

The generalized displacement δ corresponding to F is given by

$$F\delta = \sum_{j=1}^{n} F_j v_j \tag{1.181}$$

and so by (1.167), since the strain energy density is $\bar{\sigma}^2/2E$,

$$\delta = \bar{\sigma}^2 V/EF. \tag{1.182}$$

The 'stiffness' F/δ then follows from (1.180, 182) as

$$F/\delta = EV/\beta^2 l^2, \tag{1.183}$$

and shows that the best material is that with the largest E. If the weight of the structure ρV is given, then the best material is that with the largest E/ρ or the largest 'specific modulus'.

The problem of optimum elastic design in which min V is sought for given forces $\{F_j\}$ and a given displacement δ, defined by a formula like (1.181), is easily shown to lead to the same problem of optimum design, with allowable stress $\bar{\sigma}$, obtained above. The stress $\bar{\sigma}$ is now given by (1.182), where V is given by (1.179) and so

$$\bar{\sigma} = E\delta/\beta l \tag{1.184}$$

The prescribed δ must not be too great or otherwise $\bar{\sigma}$ will exceed σ and the structure will be unsafe. The general problem of optimum elastic design in which limitations are placed both upon stress and displacement presents difficulties because of the non-linearity of (1.165). Equations (1.164, 5) as well as one or more displacement restrictions have to be added to the equations of section 1.1 and this generates a problem of non-linear programming. Reference may be made to Pope and Schmit (1971) for details of numerical methods for tackling this kind of problem.

2 Beams

2.1 Beams of least volume of material

Consider the problem of designing a beam of variable section to carry given normal loads over a span from $x = 0$ to $x = l$, with given boundary conditions at the ends. Let the distributed loads be $w(x)$ per unit length and the concentrated loads be w_i at $x = x_i (i = 1, 2 \ldots n)$, together with loads at the free ends (if any). Denote the shear force by $S(x)$ and the bending moment by $M(x)$. The conditions of equilibrium require that

$$\left.\begin{aligned} &\mathrm{d}S/\mathrm{d}x + w = 0, \quad \mathrm{d}M/\mathrm{d}x + S = 0, \\ &(\Delta S)_{x=x_i} + w_i = 0 \qquad (i = 1, 2 \ldots n), \end{aligned}\right\} \tag{2.1}$$

and

$$M = 0 \text{ (pinned ends)}; \quad M = 0,\ S \text{ given (free ends)}. \tag{2.2}$$

Equations (2.1) can be integrated in the form

$$\left.\begin{aligned} S &= M_0/l - M_l/l - \mathrm{d}m/\mathrm{d}x, \\ M &= M_0(1 - x/l) + M_l x/l + m, \end{aligned}\right\} \tag{2.3}$$

where M_0, M_l are constants of integration (end moments) and $m(x)$ is the bending moment for pinned ends. Fig. 2.1 shows a graph of a typical $m(x)$ and a straight line with ordinates $-M_0(1 - x/l) - M_l x/l$. The difference in the ordinates gives $M(x)$.

Let the beam be designed according to the methods of plastic design. Denote the required limiting plastic bending moment by $M_p(x)$ and let it have a lower positive bound M_{p0}. It then follows that

$$\left.\begin{aligned} M(x) + p(x) &= M_p(x), \quad p(x) \geq 0, \\ -M(x) + q(x) &= M_p(x), \quad q(x) \geq 0, \\ M_{p0} + r(x) &= M_p(x), \quad r(x) \geq 0, \end{aligned}\right\} \tag{2.4}$$

where p, q, and r are non-negative 'slack functions'.

The relation between the area of cross-section of the beam and M_p depends upon assumptions about the nature of the section. For a rectangular section of given width the area varies as $M_p^{1/2}$. For similar sections of given shape the area varies as $M_p^{2/3}$. For a beam of 'sandwich construction', with given depth and width but variable skin thickness, the area varies as M_p. This is also true for an

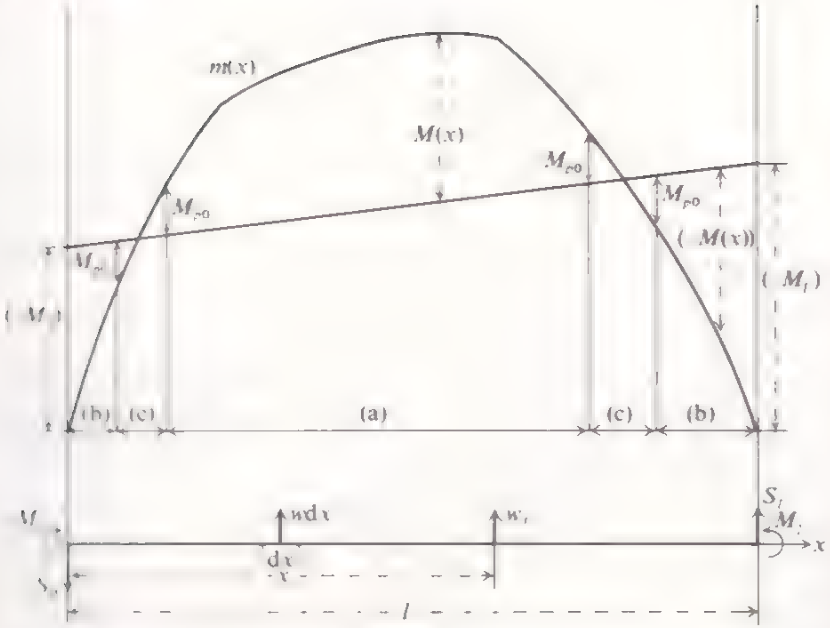


Fig. 2.1

idealized I-section of given depth and flange width but variable flange thickness. It will be assumed here that the area of the cross-section varies as M_p^α, where α is a positive number not greater than one. The condition for minimum volume of material can then be written

$$\min V = \int_0^l kM_p^\alpha \mathrm{d}x, \qquad (2.5)\dagger$$

where k is a known constant depending upon the geometry of the section and the yield stress of the material. The 'variables' for this minimization problem are the functions S and M or the constants M_0 and M_l, together with the functions p, q, r, and M_p. The 'constraints' are given by (2.1, 2, 4) or by (2.3, 4). This is a problem of the calculus of variations,‡ albeit a special one that can be readily reduced to an ordinary minimum problem.

A solution to (2.5) will be sought on the assumption that M_p is a continuous function. Since the bending moment $M(x)$ is continuous it follows by (2.4) that p, q, and r are also continuous functions.

† The case of 'cost functions' like (2.5) with $\alpha > 1$ is treated in Marcal and Prager (1964) by an ingenious but special method. The present treatment may be applied to this case as well.

‡ A good modern text is Pars (1962). The standard works do not however give much attention to problems with non-negative functions and so this subject will be dealt with in some detail here.

If $p, q, r > 0$ at a point and hence in an interval, it follows from (2.4) that M_p can be reduced in magnitude in that interval. Equation (2.5) then shows that V can be reduced. It follows that one at least of p, q, and r must be zero at every point. If either p or q is zero, then (2.4) gives $|M| = M_p \geqslant M_{p0}$ and if $r = 0$ the conclusion is $|M| \leqslant M_p = M_{p0}$. The alternatives for M_p are thus $|M|$ or M_{p0} and the greater of the two must of course be taken. The positions at which $M = \pm M_{p0}$ are shown in Fig. 2.1 and an interval (a) in which $M_p = M$, two intervals (b) in which $M_p = -M$ and two intervals (c) in which $M_p = M_{p0}$ are also indicated.

Equation (2.5) can now be expressed using (2.3) in the form

$$\begin{aligned}\min V = \int_{(a)} k\left\{M_0(1-x/l) + M_l x/l + m\right\}^\alpha \mathrm{d}x \,+ \\ + \int_{(b)} k\left\{-M_0(1-x/l) - M_l x/l - m\right\}^\alpha \mathrm{d}x + \int_{(c)} kM_{p0}^\alpha \,\mathrm{d}x,\end{aligned} \qquad (2.6)$$

where regions (a), (b), and (c) are defined by

$$\text{(a) } p = 0, q > 0, r > 0; \quad \text{(b) } p > 0, q = 0, r > 0; \quad \text{(c) } p > 0, q > 0, r = 0. \qquad (2.7)$$

It is to be remarked that (2.4) forbids $p = q = 0$, but allows $p = r = 0$ and $q = r = 0$ in regions where $m(x)$ is a linear function of x. These degenerate cases must be excluded, if $m(x)$ has such a linear part. This can be accomplished by adding a 'perturbation' to the loading, which consists of a uniform infinitesimal load η spread over the whole span. The conclusions of the present investigation can then be applied and when this has been done, the perturbation can be removed by allowing $\eta \to 0$. It is clear that in practice the perturbation has only to be carried out in imagination.

The problem of (2.6) is a minimum problem with variables M_0 and M_l. The variation of the boundaries of (a), (b), and (c) with δM_0 and δM_l produces no first order change in V, since M_p is continuous at these boundaries and so the gains and losses of the various integrals in (2.6) cancel out. The conditions for the minimum of V in (2.6) can thus be written, remembering (2.2), as

$$\left.\begin{aligned}\int_{(a)} M_p^{\alpha-1}(1-x/l)\mathrm{d}x &= \int_{(b)} M_p^{\alpha-1}(1-x/l)\mathrm{d}x \quad \text{or} \quad M_0 \text{ given,} \\ \int_{(a)} M_p^{\alpha-1}(x/l)\mathrm{d}x &= \int_{(b)} M_p^{\alpha-1}(x/l)\mathrm{d}x \quad \text{or} \quad M_l \text{ given,}\end{aligned}\right\} \qquad (2.8)$$

where

$$M_p = M \text{ in (a)}, \quad M_p = -M \text{ in (b)} \quad \text{and} \quad M_p = M_{p0} \text{ in (c)}. \qquad (2.9)$$

Equations (2.8, 9) give two simultaneous equations for M_0 and M_l and complete the solution of the present problem. A solution can be obtained by trial and

error variation of M_0 and M_l, i.e. the position of the straight line in Fig. 2.1. For a cantilever or a pin-ended beam M_0 and M_l are given by (2.2) and (2.8) is not required. For a clamped-pinned beam M_l is zero and only the first of (2.8) is needed. Both relations of (2.8) are required for a doubly-clamped beam.

The meaning of (2.8) becomes particularly simple when $\alpha = 1$. In this case

$$\left.\begin{aligned} &\int_{(a)} (l-x)\,dx = \int_{(b)} (l-x)\,dx \quad \text{or} \quad M_0 \text{ given}, \\ &\int_{(a)} x\,dx = \int_{(b)} x\,dx \quad \text{or} \quad M_l \text{ given}, \end{aligned}\right\} \tag{2.10}$$

and the equations state that the moments of the regions (a) and (b), taken about the ends $x = l$ and $x = 0$ respectively, are equal to one another. For the doubly-clamped case the lengths of (a) and (b) are equal and their centroids coincident.

2.2 An example

Consider the problem of a doubly-clamped beam loaded by a uniform normal force $-w$ per unit length acting on the intervals $0 < x < a$ and $l-a < x < l$. This gives a constant $m(x)$ for $a \leqslant x \leqslant l-a$ and thus requires the introduction of a perturbation consisting of a uniform load $-\eta$ per unit length over the whole span. The unperturbed bending moment $m(x)$ is given by

$$\left.\begin{aligned} m(x) &= wax - wx^2/2 \qquad (0 \leqslant x \leqslant a), \\ m(x) &= wa^2/2 \qquad (a \leqslant x \leqslant l/2). \end{aligned}\right\} \tag{2.11}$$

and is symmetric about $x = l/2$.

Assume for simplicity that $M_{p0} \to 0$ and that $\alpha = 1$. This means for the perturbed loading that there are no regions (c) and that (2.10) gives the points where (a) and (b) meet as $x = l/4$, $3l/4$. These values must satisfy $M(x) = 0$ and so by (2.3), since $M_0 = M_l$ by symmetry,

$$M_0 + m(l/4) = 0. \tag{2.12}$$

Equations (2.11, 12) yield, taking $\eta \to 0$,

$$\left.\begin{aligned} M_0 &= -wl(8a-l)/32 \qquad (a \geqslant l/4), \\ M_0 &= -wa^2/2 \qquad (a < l/4). \end{aligned}\right\} \tag{2.13}$$

The total bending moment $M(x) = M_0 + m(x)$ and $M_p = |M|$ for the present case. Equations (2.11, 13) thus give:

Case $a \geqslant l/4$

$$\left.\begin{aligned} M_p &= wl(8a-l)/32 - wax + wx^2/2 \qquad (0 \leqslant x \leqslant l/4), \\ M_p &= -wl(8a-l)/32 + wax - wx^2/2 \qquad (l/4 \leqslant x \leqslant a), \\ M_p &= w(4a-l)^2/32 \qquad (a \leqslant x \leqslant l/2). \end{aligned}\right\} \tag{2.14}$$

Case $a < l/4$

$$\left.\begin{aligned} M_p &= w(a-x)^2/2 \qquad (0 \leqslant x \leqslant a), \\ M_p &= 0 \qquad (a \leqslant x \leqslant l/2). \end{aligned}\right\} \tag{2.15}$$

Equations (2.14, 15) define the optimum design. In the case $a < l/4$ the beam divides into two separate cantilevers.

The volume $V_{\min}$ is given by

$$\left.\begin{aligned} V_{\min} &= wk(l^3 - 12la^2 + 48la^2 - 32a^3)/96 \qquad (a \geqslant l/4), \\ V_{\min} &= wka^3/3 \qquad (a < l/4). \end{aligned}\right\} \tag{2.16}$$

2.3 Application of the calculus of variations

The conclusions of Section 2.1 may be obtained by a direct use of the calculus of variations. This alternative approach, although much less simple than that of Section 2.1, does throw new light on the problem. It also provides the opportunity to rehearse the techniques of this calculus, making use of a simple problem whose solution is already known and which is capable of direct graphical representation (Fig. 2.1).

It is necessary first of all, as it was in Section 2.1, to remove the degenerate cases $p = r = 0$ and $q = r = 0$ by adding a uniform infinitesimal loading η. It is also necessary, for the special case when $M_{p0} = 0$, to remove the even more degenerate case $p = q = r = 0$. This can be done by introducing a positive M_{p0} and letting $M_{p0} \to 0$ after the analysis is complete (cf. Section 2.2).

The minimum solution must consist, after (2.7), of regions (a), (b) and (c). It will be assumed that the span of the beam can be divided into a finite number of intervals for which one or other of these regimes is valid.

Consider now 'variations' $\delta S(x)$, $\delta M(x)$, $\delta p(x)$, $\delta q(x)$, $\delta r(x)$ and $\delta M_p(x)$ from an optimum solution $S(x)$, $M(x)$, $p(x)$, $q(x)$, $r(x)$ and $M_p(x)$, which is assumed to exist. These must satisfy, by (2.1, 2, 4),

$$d\delta S/dx = 0, \quad d\delta M/dx + \delta S = 0, \quad \delta S \text{ continuous}, \tag{2.17}$$

$$\delta M = 0 \text{ (pinned or free ends)}, \quad \delta S = 0 \text{ (free ends)}, \tag{2.18}$$

$$\delta M + \delta p = \delta M_p, \quad -\delta M + \delta q = \delta M_p, \quad \delta r = \delta M_p, \tag{2.19}$$

and

$$\delta p \geqslant 0\ (p = 0), \quad \delta q \geqslant 0\ (q = 0), \quad \delta r \geqslant 0\ (r = 0). \tag{2.20}$$

They must also imply, by (2.5), that

$$\delta V = \int_0^l \alpha k M_p^{\alpha-1}\, \delta M_p\, dx \geqslant 0. \tag{2.21}$$

The next step is to solve (2.17, 19) and thus to divide the variations into 'dependent' and 'independent' variations. The dependent variations must be such

that no additional restriction is placed upon their values and they are thus able to take any values the independent variations impose upon them. The independent variations must include all those which are subject to restrictions like (2.18, 20). A problem in which this division of the variations can be carried out is a 'regular' problem, otherwise it is 'irregular'. The reason for the exclusion of degenerate cases, in which two or three of p, q, and r are zero, is that (2.20) requires that two or three of the corresponding variations must be taken as independent. It is easily seen that this is not possible.

Equations (2.17) can be solved to give

$$\delta S = \delta M_0/l - \delta M_l/l, \quad \delta M = \delta M_0(1 - x/l) + \delta M_l x/l, \tag{2.22}$$

where δM_0 and δM_l are variations of the constants M_0 and M_l of (2.3). These last may be restricted by (2.18) and should be taken as independent. Equation (2.22) shows that this is possible since δS and δM have no restrictions placed upon them, in the open interval $(0, l)$ at any rate. Equations (2.19) may be solved for δM_p and for two of δp, δq, and δr in terms of the remaining variation of a slack function and, after (2.22), in terms of δM_0 and δM_l. This is satisfactory for each of the regions (a), (b), and (c), since by (2.7) only one of p, q, and r vanishes and so by (2.20) only one of the corresponding variations need be independent. The scheme for the solution of (2.19, 22) is thus

Region	Dependent variations	Independent variations
(a)	$\delta S,\ \delta M,\ \delta M_p,\ \delta q,\ \delta r$	$\delta M_0,\ \delta M_l,\ \delta p$
(b)	$\delta S,\ \delta M,\ \delta M_p,\ \delta p,\ \delta r$	$\delta M_0,\ \delta M_l,\ \delta q$
(c)	$\delta S,\ \delta M,\ \delta M_p,\ \delta p,\ \delta q$	$\delta M_0,\ \delta M_l,\ \delta r$

(2.23)

It is unnecessary to write the actual solutions.

Lagrangian 'multipliers' must now be introduced just as in (1.152). It is convenient to introduce the constant k as a factor, to introduce a positive infinitesimal ϵ as a divisor and to write the multipliers corresponding to the first two of (2.17) and to (2.19) as $kv(x)/\epsilon$, $k\theta(x)/\epsilon$, $k\kappa'(x)/\epsilon$, $k\kappa''(x)/\epsilon$ and $k\kappa_0(x)/\epsilon$. Equations (2.17, 19, 21) then give

$$\int_0^l \Big\{\epsilon\alpha M_p^{\alpha-1}\,\delta M_p + v\mathrm{d}\delta S/\mathrm{d}x + \theta(\mathrm{d}\delta M/\mathrm{d}x + \delta S) + \kappa'(\delta M + \delta p - \delta M_p) + \kappa''(-\delta M + \delta q - \delta M_p) + \kappa_0(\delta r - \delta M_p)\Big\}\,\mathrm{d}x \geqslant 0. \tag{2.24}$$

Integrating by parts, assuming v and θ continuous, and collecting like terms gives

$$\int_0^l \Big\{(\epsilon\alpha M_p^{\alpha-1} - \kappa' - \kappa'' - \kappa_0)\delta M_p + (\theta - \mathrm{d}v/\mathrm{d}x)\delta S + (\kappa' - \kappa'' - \mathrm{d}\theta/\mathrm{d}x)\delta M + \kappa'\delta p + \kappa''\delta q + \kappa_0\delta r\Big\}\,\mathrm{d}x + [v\delta S + \theta\delta M]_0^l \geqslant 0, \tag{2.25}$$

which is valid for all variations satisfying (2.17, 19, 21).

The Lagrangian multipliers must now be chosen so as to remove from (2.25) the dependent variations of (2.23). This requires

$$\kappa' + \kappa'' + \kappa_0 = \epsilon\alpha M_p^{\alpha-1}, \tag{2.26}$$

$$\mathrm{d}v/\mathrm{d}x = \theta, \quad \mathrm{d}\theta/\mathrm{d}x = \kappa' - \kappa'' = \kappa, \tag{2.27}$$

and

$$\kappa' = 0\ (p > 0), \quad \kappa'' = 0\ (q > 0), \quad \kappa_0 = 0\ (r > 0), \tag{2.28}$$

where a new function κ has been introduced. Equations (2.26, 28) can always be solved for κ', κ'' and κ_0 in each region (a), (b), and (c). Equations (2.27) can then be integrated over the whole span to yield continuous functions v and θ. The required choice for the Langrangian multipliers is thus possible.

Imposing (2.26–28) on (2.25) then gives, using (2.22) for the integrated terms,

$$\int_{(a)} \epsilon\alpha M_p^{\alpha-1}\,\delta p\,\mathrm{d}x + \int_{(b)} \epsilon\alpha M_p^{\alpha-1}\,\delta q\,\mathrm{d}x + \int_{(c)} \epsilon\alpha M_p^{\alpha-1}\,\delta r\,\mathrm{d}x + [\{v(l) - v(0)\}/l - \theta(0)]\delta M_0 - [\{v(l) - v(0)\}/l - \theta(l)]\delta M_l \geqslant 0, \tag{2.29}$$

where all the variations are independent and can be given arbitrary values, subject to (2.18, 20). This means that each term in (2.29) must be non-negative, since the other terms can be set equal to zero. The first, second, and third terms of (2.29) are non-negative, thanks to (2.20). The last two terms can be zero if M_0 or M_l are given by (2.2), but otherwise they can always be made negative, unless the coefficients of δM_0 and δM_l are zero. This gives

$$\left.\begin{aligned} \{v(l) - v(0)\}/l &= \theta(0) \quad \text{or} \quad M_0 \text{ given,} \\ \{v(l) - v(0)\}/l &= \theta(l) \quad \text{or} \quad M_l \text{ given.} \end{aligned}\right\} \tag{2.30}$$

Equation (2.27) gives

$$\left.\begin{aligned} \int_0^l \kappa(1 - x/l)\,\mathrm{d}x &= -\theta(0) + \{v(l) - v(0)\}/l, \\ \int_0^l \kappa(x/l)\,\mathrm{d}x &= \theta(l) - \{v(l) - v(0)\}/l, \end{aligned}\right\} \tag{2.31}$$

and (2.26, 28) yield

$$\kappa = \epsilon\alpha M_p^{\alpha-1} \text{ in (a)}, \quad \kappa = -\epsilon\alpha M_p^{\alpha-1} \text{ in (b)}, \quad \kappa = 0 \text{ in (c)}. \tag{2.32}$$

Substitution from (2.32) into (2.31) and from thence into (2.30) then gives (2.8) once more and the present investigation agrees with that of Section 2.1.

The Lagrangian multipliers can be given a kinematic interpretation. If v is a virtual normal displacement, then by (2.27) θ is the corresponding virtual rotation and κ the corresponding virtual curvature of the beam. Equations (2.30) then give, on removing a rigid body motion by writing $v(0) = v(l) = 0$,

$$\theta(0) = 0 \text{ or } M_0 = 0, \quad \theta(l) = 0 \text{ or } M_l = 0, \tag{2.33}$$

which are the usual kinematic boundary conditions for the doubly-clamped and clamped-pinned cases.

In the special case when $M_{p0} \to 0$, the solutions obtained above can be treated as optimum elastic designs. Regions (c) are now absent and in regions (a) and (b) the curvature κ is given by (2.32), while the corresponding bending moments are $M = \pm M_p$. It follows that

$$M/\kappa = M_p^{2-\alpha}/\epsilon\alpha. \tag{2.34}$$

For sections, which have second moments of area I given by

$$I = k_1 M_p^{2-\alpha}, \tag{2.35}$$

where k_1 is a constant, it is only necessary to write

$$\epsilon = 1/E\alpha k_1, \tag{2.36}$$

where E is Young's modulus, to convert (2.34) into the usual bending moment-curvature relation for the elastic deformation of beams and hence to allow $v(x)$ to be identified with the real displacement. The relation (2.35) is valid for rectangular beams of constant width ($\alpha = 1/2$), similar sections of varying size ($\alpha = 2/3$) and for sandwich beams and idealized I-sections ($\alpha = 1$). For the elastic case M_p should be taken as the bending moment at which plastic flow begins, with k of (2.5) defined to suit.

2.4 Use of the Lagrangian

The results of Section 2.3 can be obtained, formally at any rate, by constructing a Lagrangian in a manner analogous to that used to form (1.62), and then writing the conditions that this Lagrangian should have a minimum.

The Lagrangian V^* for the problem of (2.1, 2, 4, 5) can be written

$$\epsilon V^* = k\int_0^l \Big\{\epsilon M_p^\alpha + v(\mathrm{d}S/\mathrm{d}x + w) + \theta(\mathrm{d}M/\mathrm{d}x + S) + \kappa'(M + p - M_p) + \\ + \kappa''(-M + q - M_p) + \kappa_0(M_{p0} + r - M_p)\Big\}\,\mathrm{d}x + \sum_{i=1}^{n} v_i\Big\{(\Delta S)_{x=x_i} + w_i\Big\}, \tag{2.37}$$

where $v(x)$, $\theta(x)$, $\kappa'(x)$, $\kappa''(x)$, $\kappa_0(x)$ and $v_i (i = 1, 2 \ldots n)$ are Lagrangian multipliers and ϵ the positive infinitesimal introduced before. Conditions are to be sought for min V^*, subject to the boundary conditions (2.2)† and the non-negativity conditions for p, q, and r.

Integration of (2.37) by parts gives

$$\epsilon V^* = k\int_0^l \Big\{\epsilon M_p^\alpha - (\kappa' + \kappa'' + \kappa_0)M_p + (\theta - \mathrm{d}v/\mathrm{d}x)S + (\kappa' - \kappa'' - \mathrm{d}\theta/\mathrm{d}x)M + \\ + \kappa' p + \kappa'' q + \kappa_0 r + vw + \kappa_0 M_{p0}\Big\}\,\mathrm{d}x + k\sum_{i=1}^{n}\Big[\Big\{v_i - v(x_i)\Big\}(\Delta S)_{x=x_i} + v_i w_i\Big] + \\ + k[vS + \theta M]_0^l, \tag{2.38}$$

and so the condition for V^* to be a minimum can be written

$$\int_0^l \Big\{(\epsilon\alpha M_p^{\alpha-1} - \kappa' - \kappa'' - \kappa_0)\delta M_p + (\theta - \mathrm{d}v/\mathrm{d}x)\delta S + (\kappa' - \kappa'' - \mathrm{d}\theta/\mathrm{d}x)\delta M + \\ + \kappa'\delta p + \kappa''\delta q + \kappa_0\delta r\Big\}\,\mathrm{d}x + \sum_{i=1}^{n}\Big\{v_i - v(x_i)\Big\}(\Delta\delta S)_{x=x_i} + [v\delta S + \theta\delta M]_0^l \geqslant 0. \tag{2.39}$$

The variations in (2.39) can be taken as arbitrary apart from δp, δq, and δr, which must satisfy (2.20) and δS, δM at the ends, which by (2.2) will be zero, when not arbitrary. This means that each term in (2.39) involving a variation as a multiplier in its integrand must be non-negative and indeed the integrands of such terms must themselves be non-negative for all x. It follows that the factors multiplying completely arbitrary variations must vanish and those which multiply non-negative variations, like δp for $p = 0$, must themselves be non-negative. Application of these rules to (2.39) gives

$$\left.\begin{array}{l} \kappa' + \kappa'' + \kappa_0 = \epsilon\alpha M_p^{\alpha-1}, \\ \mathrm{d}v/\mathrm{d}x = \theta, \quad \mathrm{d}\theta/\mathrm{d}x = \kappa' - \kappa'', \\ \kappa' = 0\ (p > 0), \quad \kappa' \geqslant 0\ (p = 0), \\ \kappa'' = 0\ (q > 0), \quad \kappa'' \geqslant 0\ (q = 0), \\ \kappa_0 = 0\ (r > 0), \quad \kappa_0 \geqslant 0\ (r = 0), \\ v_i = v(x_i) \qquad (i = 1, 2 \ldots n), \\ \quad v = 0 \text{ or } S \text{ given}; \quad \theta = 0 \text{ or } M \text{ given (at ends } x = 0, l). \end{array}\right\} \tag{2.40}$$

† These can be included when well defined. If $M_l = 0$, a term $\theta_l M_l$ can be added to (2.37), where θ_l is a constant multiplier.

For the regions defined by (2.7) the equations (2.40) give the same results as those of Section 2.3. The first three of (2.40) agree with (2.26, 27) and the conditions for p, q, and $r > 0$ in (2.40) and (2.28) are the same. It follows that the values of κ', κ'', and κ_0 agree in the regions (a), (b), and (c) and that (2.32) follows from (2.40). The boundary conditions (2.33) also follow from the last line of (2.40). Equations (2.8) can thus be derived just as they were in Section 2.3. The necessary conditions for the minimum of (2.5), subject to (2.1, 2, 4), can thus be obtained by writing the necessary conditions for the minimum of the Lagrangian (2.37), subject to (2.2) and p, q, and $r \geqslant 0$. This conclusion has been established on the assumption that the minimum solution is non-degenerate. However the results of (2.40) are valid for degenerate cases as well, when properly interpreted, just as were the results of Sections 2.1, 3.

2.5 Dual form of the linear problem

The Lagrangian of (2.37) becomes a linear functional when $\alpha = 1$ and for this special case a theory of duality can be developed, very similar to that for linear programming in Section 1.4.

The dual problem corresponding to the primal problem of (2.1, 2, 4, 5) can be obtained from the Lagrangian (2.37) by imposing the conditions (2.40) as constraints.† The resulting problem will be shown to be a maximum problem and can be obtained by substituting from (2.40) into (2.38) and by eliminating p, q, r, S, and M from (2.40). This gives the dual problem in the form:

$$\max W = (k/\epsilon)\left\{\int_0^l (wv + M_{p0}\kappa_0)\,\mathrm{d}x + \sum_{i=1}^{n} w_i v(x_i)\right\}, \tag{2.41}$$

subject to,

$$\left.\begin{aligned} &\kappa' + \kappa'' + \kappa_0 = \epsilon, \\ &\kappa',\ \kappa'',\ \kappa_0 \geqslant 0, \\ &\mathrm{d}v/\mathrm{d}x = \theta, \quad \mathrm{d}\theta/\mathrm{d}x = \kappa' - \kappa'', \\ &v = 0 \text{ at a support}, \quad \theta = 0 \text{ at a clamp}. \end{aligned}\right\} \tag{2.42}$$

Let κ', κ'', κ_0, v and θ be a 'feasible solution' of the dual, i.e. any solution

† See the discussion of (1.62) above.

of (2.42). Let S, M, p, q, r, and M_p be a feasible solution of the primal, i.e. any solution of (2.1, 2, 4). For these

$$\begin{aligned} W &= (k/\epsilon)\left\{\int_0^l (-v\,\mathrm{d}S/\mathrm{d}x + M_{p0}\kappa_0)\,\mathrm{d}x + \sum_{i=1}^{n} w_i v(x_i)\right\}, \text{ by (2.41) and (2.1),} \\ &= (k/\epsilon)\int_0^l (\theta S + M_{p0}\kappa_0)\,\mathrm{d}x, \text{ by (2.1), (2.2), and (2.42),} \\ &= (k/\epsilon)\int_0^l (-\theta\,\mathrm{d}M/\mathrm{d}x + M_{p0}\kappa_0)\,\mathrm{d}x, \text{ by (2.1),} \\ &= (k/\epsilon)\int_0^l \left\{M(\kappa' - \kappa'') + M_{p0}\kappa_0\right\}\mathrm{d}x, \text{ by (2.2) and (2.42),} \\ &= (k/\epsilon)\int_0^l \left\{(M_p - p)\kappa' + (M_p - q)\kappa'' + (M_p - r)\kappa_0\right\}\mathrm{d}x, \text{ by (2.4),} \\ &= V - (k/\epsilon)\int_0^l (p\kappa' + q\kappa'' + r\kappa_0)\,\mathrm{d}x, \text{ by (2.42) and (2.5),} \\ &\leqslant V, \text{ by (2.4) and (2.42).} \end{aligned} \tag{2.43}$$

It follows that,

$$W \leqslant V_{\min} \leqslant V, \tag{2.44}$$

which shows that W is bounded above and that upper and lower bounds can be found for $V_{\min}$ by using feasible solutions to the primal and dual respectively.

If it is now assumed, as in Section 2.3, that the primal problem has an optimum solution, then feasible solutions to the primal and dual problems can be found, which satisfy (2.40), the necessary conditions for a minimum of the primal. These give by (2.40, 43) $W = V$ and so by (2.44) $W = V_{\min}$. This W is also $W_{\max}$, as follows from (2.44), and so the dual has an optimum solution with

$$W_{\max} = V_{\min}. \tag{2.45}$$

The volume of material in the optimum beam can thus be found by solving the dual problem. The present discussion can be compared with that of Section 1.4.

Use of the kinematic interpretation of the multipliers allows, when $M_{p0} \to 0$ at any rate, the interpretation of W as the virtual work of the given loads taken over the virtual displacements v. Equation (2.42) imposes the restriction that the absolute magnitude of the virtual curvature $|\kappa' - \kappa''|$ may not exceed ϵ.

Necessary and sufficient conditions for an optimum design can be formulated for beams, just as in (1.70) for frameworks. The present duality theory gives the following result:

A distribution of M_p gives an optimum design, if $M_p = |M|$ for $|M| > M_{p0}$ and $M_p = M_{p0}$ for $|M| < M_{p0}$, where M is a possible statically correct bending moment

distribution, and if there is a virtual displacement v, which satisfies the kinematic boundary conditions and has a virtual curvature ϵ when $M = M_p$, a virtual curvature $-\epsilon$ when $M = -M_p$ and a virtual curvature of zero when $|M| < M_{p0}$. (2.46)

This is a Michell theorem for beams like that of (1.70) for frameworks.

2.6 Beams of maximum stiffness

The present problem, like that of Section 1.12, will be interpreted as meaning the determination of that structure of given volume, which stores, for given loads, the least amount of strain energy U. For a beam, loaded as in Section 2.1, this means

$$\min 2U = \int_0^l wv\,dx + \sum_{i=1}^{n} w_i v(x_i) + [Sv]_0^l, \tag{2.47}$$

where $v(x)$ is now the real displacement of the beam and the last term has been included to allow for the case of loaded free end.

The second moment of area I is assumed to be related to the area of the cross-section A by

$$A = k_0 I^\beta \tag{2.48}$$

where k_0 is a given constant and β a positive number not greater than one. For a rectangular section of given width $\beta = 1/3$, for similar sections $\beta = 1/2$ and for a sandwich beam or an idealized I-section $\beta = 1$. The condition of given volume V can thus be written

$$\int_0^l I^\beta\,dx = V/k_0. \tag{2.49}$$

The equations governing the deformation of the beam are

$$\left.\begin{aligned} &dS/dx + w = 0,\ dM/dx + S = 0,\ (\Delta S)_{x=x_i} + w_i = 0 \quad (i = 1, 2 \ldots n),\\ &M = EI\kappa,\ \ \kappa = d\theta/dx,\ \ \theta = dv/dx,\\ &S \text{ given},\ M = 0 \text{ (free end)};\ \ v = M = 0 \text{ (pinned end)};\ \ v = \theta = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(clamped end).}\end{aligned}\right\} \tag{2.50}$$

A positive lower limit I_0 is placed on I by the relation

$$I_0 + r = I, \quad r \geqslant 0. \tag{2.51}$$

The problem of maximum stiffness for a beam is thus to find the minimum U of (2.47), subject to (2.49, 50, 51). The design is determined by $I(x)$ and the minimum problem involves in addition the functions S, M, κ, θ, v, and r. The methods used in Section 2.3 may be applied to the equations as they stand, but it is easier to reduce the size of the problem first.

Use of (2.50) or of a standard formula enables (2.47) to be replaced by

$$\min 2U = \int_0^l (M^2/EI)\,dx, \tag{2.52}$$

where, again by (2.50) or by (2.1, 2), M is expressed in terms of M_0 and M_l by (2.3). The values of M_0 and M_l for a given $I(x)$ follow from (2.50), using (2.31), or more usefully, using Castigliano's theorem, from

$$\partial U/\partial M_0 = 0 \quad \text{or} \quad M_0 \text{ given}, \qquad \partial U/\partial M_l = 0 \quad \text{or} \quad M_l \text{ given}. \tag{2.53}$$

Let $I(x)$ be varied to $I(x) + \delta I(x)$. This will induce changes in M_0 and M_l, unless one or both of them are given. Such changes themselves will not affect U in (2.52), thanks to (2.53). The condition that (2.52) gives a minimum can thus be expressed by

$$\int_0^l (-M^2/EI^2)\,\delta I\,dx \geqslant 0, \tag{2.54}$$

or, using (2.50), by

$$\int_0^l \kappa^2\,\delta I\,dx \leqslant 0. \tag{2.55}$$

Equation (2.55) will be valid, by (2.49) and (2.51), for variations which satisfy

$$\int_0^l I^{\beta-1}\delta I\,dx = 0, \quad \delta I = \delta r, \quad \delta r \geqslant 0 \ \ (r = 0). \tag{2.56}$$

The span of the beam can be divided into two regions defined by (a) $r > 0$ and (b) $r = 0$. Equations (2.55, 56), with δI eliminated, can then be written

$$\left.\begin{aligned} &\int_{(a)} \kappa^2 \delta r\,dx + \int_{(b)} \kappa^2 \delta r\,dx \leqslant 0,\\ &\text{when}\\ &\int_{(a)} I^{\beta-1}\delta r\,dx + \int_{(b)} I_0^{\beta-1}\delta r\,dx = 0 \quad \text{and} \quad \delta r \geqslant 0 \text{ in (b).}\end{aligned}\right\} \tag{2.57}$$

Taking $\delta r = 0$ in (b) and since δr is not restricted as to sign in (a), it follows that

$$\int_{(a)} \kappa^2 \delta r\,dx = 0, \quad \text{when} \quad \int_{(a)} I^{\beta-1}\delta r\,dx = 0. \tag{2.58}$$

This implies†

$$\kappa^2 = CI^{\beta-1}, \text{ in (a)}, \tag{2.59}$$

where C is a positive constant. Substituting from (2.59) in (2.57) gives

$$\int_{(b)} (CI_0^{\beta-1} - \kappa^2)\delta r \, dx \geq 0 \tag{2.60}$$

and so, since by (2.57) $\delta r \geq 0$ in (b),

$$\kappa^2 \leq CI_0^{\beta-1}, \text{ in (b)}. \tag{2.61}$$

Equations (2.59, 61) combine to give, recalling (2.51),

$$\left.\begin{aligned} \kappa^2 &= CI^{\beta-1} \quad (I > I_0), \\ \kappa^2 &\leq CI^{\beta-1} \quad (I = I_0). \end{aligned}\right\} \tag{2.62}$$

The solution to the present problem is thus given by (2.50, 62), with the constant C determined by (2.49).

In the case when $I_0 \to 0$, equations (2.48, 62) give

$$EI\kappa^2/2A = EC/2k_0, \tag{2.63}$$

which says that the average density of strain energy at a section of the beam is the same for all sections. This rule determines the design of a beam of maximum stiffness. Exactly the same rule can be established for the elastic design of beams of minimum volume of material considered in Section 2.3. The area of section was written there as kM_p^α and I was given by (2.35). The bending moment M was always $\pm M_p$ and so

$$M^2/2EIA = 1/2Ek_1k, \tag{2.64}$$

which again gives constant average strain energy at a section. The two kinds of optimum design are thus identical in form. The volume can be made identical (equal to V of (2.49)) by an adjustment of k in (2.5).‡ The overall displacement follows from the total strain energy U, which by (2.64) is given by

$$U = V/2Ek_1k, \tag{2.65}$$

where an adjustment must be made to k_1, corresponding to that made to k.§

† This is a standard lemma of the calculus of variations. Equations (2.58) imply $\int_{(a)} (\kappa^2/I^{\beta-1} - C)I^{\beta-1}\delta r \, dx = 0$, with $I^{\beta-1}\delta r = \kappa^2/I^{\beta-1} - C$ and the constant C determined by $\int_{(a)} (\kappa^2/I^{\beta-1} - C)\, dx = 0$. It follows that $\int_{(a)} (\kappa^2/I^{\beta-1} - C)^2 \, dx = 0$ and hence (2.59).

‡ k can be varied by varying the assumed yield stress.

§ If k is multiplied by ρ, then $k_1 k$ must be multiplied by $\rho^{2/\alpha}$.

2.7 Pin-ended strut of least volume of material

Consider the problem of designing a pin-ended strut, with the least volume of material, to carry an end load P over a span l. Let the relation between the least second moment of area I of the section and the section area A be given by (2.48). The objective can thus be expressed by

$$\min V = \int_0^l k_0 I^\beta \, dx. \tag{2.66}$$

The design of the strut is defined by $I(x)$ and, corresponding to any such design, there will be a least critical load or Euler buckling load P_E with its mode of buckling defined by a displacement $v(x)$. The following equations will be satisfied:

$$EI\, d^2v/dx^2 + P_E v = 0, \tag{2.67}$$

$$v(0) = v(l) = 0, \tag{2.68}$$

where the solution with the smallest P_E is to be taken. The function $v(x)$ can be fixed completely, apart from sign, by imposing the condition

$$\int_0^l (dv/dx)^2 \, dx = 1. \tag{2.69}$$

Equations (2.67–69) give the Rayleigh formula

$$P_E = \int_0^l EI(d^2v/dx^2)^2 \, dx. \tag{2.70}$$

The safety of the strut is ensured by

$$P + p = P_E, \qquad p \geq 0. \tag{2.71}$$

However if $p > 0$, a design, with I replaced by PI/P_E, which by (2.67–69) will have the same mode of buckling $v(x)$ and a critical load P, will give (2.66) a smaller value of V. It is therefore possible, for the present purpose, to replace (2.71) by

$$P = P_E. \tag{2.72}$$

A variation δI from the minimum solution must satisfy, by (2.66, 72)

$$\int_0^l I^{\beta-1}\delta I \, dx \geq 0, \quad \text{when } \delta P_E = 0. \tag{2.73}$$

A change of I to $I + \delta I$ will produce a corresponding change in buckling mode. Such a change will however only induce a second order change in δP_E (Rayleigh's principle), since P_E is the minimum of the integral of (2.70) subject to (2.68, 69).

It results that the condition of (2.73) may be written

$$\delta P_E = \int_0^l E\delta I(\mathrm{d}^2v/\mathrm{d}x^2)^2\,\mathrm{d}x = 0. \tag{2.74}$$

Assuming that for a minimum†

$$I(x) > 0 \qquad (0 < x < l), \tag{2.75}$$

it follows, since δI may have its sign reversed without becoming inadmisible, that

$$\int_0^l I^{\beta-1}\,\delta I\,\mathrm{d}x = 0, \quad \text{when} \int_0^l E\,\delta I(\mathrm{d}^2v/\mathrm{d}x^2)^2\,\mathrm{d}x = 0. \tag{2.76}$$

This is of the same form as (2.58) and so yields, as in (2.59), what is in fact the same relationship,

$$(\mathrm{d}^2v/\mathrm{d}x^2)^2 = CI^{\beta-1} \qquad (0 < x < l). \tag{2.77}$$

The solution to the present problem is thus governed by (2.67–69, 72, 77).

Equations (2.67, 77) give

$$I = C_1 v^{2/(\beta+1)}, \tag{2.78}$$

where C_1 is a positive constant and v is taken with a positive sign.† Equations (2.67, 72, 78) give

$$\mathrm{d}^2v/\mathrm{d}x^2 + (P/EC_1)v^{(\beta-1)/(\beta+1)} = 0. \tag{2.79}$$

The integral of (2.79), which satisfies (2.68), can be written in the parametric form

$$x = lF(\theta,\beta)/F(\pi,\beta),\;\; v = \{l/F(\pi,\beta)\}^{\beta+1}\{P\beta/EC_1(\beta+1)\}^{(\beta+1)/2}(\sin\theta)^{(\beta+1)/\beta}, \tag{2.80}$$

where

$$F(\theta,\beta) = \int_0^\theta (\sin\theta)^{1/\beta}\,\mathrm{d}\theta. \tag{2.81}$$

The constant C_1 may be chosen to satisfy (2.69), but there is no need to do this, since it does not affect the design. Equations (2.78, 80) give

$$I = \{l/F(\pi,\beta)\}^2\,\{P\beta/E(\beta+1)\}\,(\sin\theta)^{2/\beta}, \tag{2.82}$$

and hence by (2.66)

$$V_{\min} = k_0\{l/F(\pi,\beta)\}^{2\beta+1}\,\{P\beta/E(\beta+1)\}^\beta\,F\{\pi,\beta/(2\beta+1)\}. \tag{2.83}$$

† A zero of $I(x)$ not at an end will introduce a pinned joint at the same place. It also gives, by (2.67), $v = 0$ which cannot give minimum P_E.

The following results derived from (2.81) are required for applications:

$$\left.\begin{aligned} &F(\theta,1) = 1-\cos\theta,\quad F(\pi,1) = 2,\\ &F(\theta,1/2) = \theta/2 - (1/4)\sin 2\theta,\quad F(\pi,1/2) = \pi/2,\\ &F(\theta,1/3) = 2/3 - \cos\theta + (1/3)\cos^3\theta,\quad F(\pi,1/3) = 4/3,\\ &F(\pi,1/4) = 3\pi/8,\quad F(\pi,1/5) = 16/15. \end{aligned}\right\} \tag{2.84}$$

Equations (2.83, 84) give for the minimum volumes:

$$\left.\begin{aligned} &V = (1/12)k_0(P/E)l^3 && (\beta = 1,\ \text{sandwich beam of given overall dimensions}),\\ &V = (\sqrt{3}/2\pi)k_0(P/E)^{1/2}l^2 && (\beta = 1/2,\ \text{similar sections of varying size}),\\ &V = (3^{2/3}/5)k_0(P/E)^{1/3}l^{5/3} && (\beta = 1/3,\ \text{rectangular section of constant width}), \end{aligned}\right\} \tag{2.85}$$

which may be compared with the result for a uniform strut, given by

$$V = k_0(1/\pi^{2\beta})(P/E)^\beta l^{2\beta+1}. \tag{2.86}$$

The volume ratios for $\beta = 1, 1/2, 1/3$ respectively are $\pi^2/12 \simeq 0{\cdot}82$, $\sqrt{3}/2 \simeq 0{\cdot}87$ and $(3\pi)^{2/3}/5 \simeq 0{\cdot}89$. These give a measure for the saving of material obtained by tapering a strut.†

2.8 Structures built from beams

Examples of structures built only from beams are those stiff-jointed plane frameworks, whose stability derives from their stiff joints, and grillages made up of beams with low torsional stiffness. If, for simplicity, each component beam is assumed to have a uniform section, the optimum design of such structures reduces itself to a problem of linear programming, when a cost function is used, which is linear in the unknown limiting plastic moments. The two kinds of structures present identical mathematical problems and the treatment given here is applicable to both.

Consider a structure of given layout loaded by given concentrated forces $F_i(l = 1, 2 \ldots n)$. Each portion of beam between nodes has a uniform section and the 'design' of the structure is determined by the limiting plastic moments $M_{pi}(i = 1, 2 \ldots m)$ of these beam elements, which are m in number. The 'cost' of the beams is assumed to be jointly proportional to their M_{pi}‡ and their lengths $l_i(i = 1, 2 \ldots m)$. The objective can thus be formulated as

$$\min V = \sum_{i=1}^m M_{pi}l_i. \tag{2.87}$$

† This is by no means a new problem. The result for $\beta = 1/2$ was given by Clausen at St. Petersburg in 1851. A recent paper which places lower bounds on I is Trahair and Booker (1970).

‡ Costs proportional to a power of M_{pi} are considered in Prager (1956).

The method of plastic or limit design will be used. This requires the consideration of all possible plastic hinges, those at the ends of each member and those at internal points where forces are applied. Since it is not known in advance which M_{pi} at a node is the least, hinges must be given to each member at that node. The structure becomes a mechanism, when the hinges are introduced. Let the number of degrees of freedom be f and let ϕ_k $(k = 1, 2 \,..\, f)$ be a set of Lagrangian coordinates, which define completely the 'modes of collapse' of the mechanism. Denote by $\theta_j (j = 1, 2 \,..\, h)$ the rotations of the hinges and by $u_l (l = 1, 2 \,..\, n)$ the displacements corresponding to the forces $[F_l]$. The geometry of the structure determines the relations

$$\theta_j = \sum_{k=1}^{f} T_{jk}\phi_k \quad (j = 1, 2 \,..\, h), \quad u_l = \sum_{k=1}^{f} U_{lk}\phi_k \quad (l = 1, 2 \,..\, n), \tag{2.88}$$

where $[T_{jk}]$ and $[U_{lk}]$ are known matrices.

If $M_j (j = 1, 2 \,..\, h)$ are the bending moments at the positions of the hinges, which correspond to $\{\theta_j\}$, then the principle of virtual work gives

$$\sum_{j=1}^{h} M_j\theta_j = \sum_{l=1}^{n} F_l u_l. \tag{2.89}$$

Substitution from (2.88) into (2.89) yields

$$\sum_{j=1}^{h} M_j \sum_{k=1}^{f} T_{jk}\phi_k = \sum_{l=1}^{n} F_l \sum_{k=1}^{f} U_{lk}\phi_k, \tag{2.90}$$

which is true for arbitrary $\{\phi_k\}$. The equations of equilibrium for the moments $[M_j]$ are thus

$$\sum_{j=1}^{h} M_j T_{jk} = \sum_{l=1}^{n} F_l U_{lk} \qquad (k = 1, 2 \,..\, f). \tag{2.91}$$

The safety of the structure requires, in addition to satisfying (2.91), that the moments $[M_j]$ are limited by

$$|M_j| \leqslant \sum_{i=1}^{m} \mu_{ji} M_{pi} \qquad (j = 1, 2 \,..\, h), \tag{2.92}$$

where μ_{ji} takes the value unity if the jth hinge lies on the ith member, but is otherwise zero. The limiting plastic moments are assumed to be bounded below or that

$$M_{pi} \geqslant M_{p0} > 0 \qquad (i = 1, 2 \,..\, m). \tag{2.93}$$

The problem of (2.87, 91–93) is a problem of linear programming.† Introducing slack variables p_j and $q_j (j = 1, 2 \,..\, h)$ in (2.92) and surplus variables $r_i (i = 1, 2 \,..\, m)$ in (2.93) puts the constraints into the equation form

$$\left.\begin{aligned} M_j + p_j = \sum_{i=1}^{m} \mu_{ji} M_{pi}, \; -M_j + q_j = \sum_{i=1}^{m} \mu_{ji} M_{pi}, \; &\text{where } p_j, q_j \geqslant 0 \; (j = 1, 2 \,..\, h), \\ M_{pi} = M_{p0} + r_i, \quad &\text{where } r_i \geqslant 0 \; (i = 1, 2 \,..\, m). \end{aligned}\right\} \tag{2.94}$$

The conditions for minimum V of (2.87) and the dual problem are most readily obtained by constructing a Lagrangian as in (1.62). Introducing Lagrangian multipliers $\phi_k (k = 1, 2 \,..\, f)$, $\theta'_j, \theta''_j (j = 1, 2 \,..\, h)$ and $\alpha_i (i = 1, 2 \,..\, m)$, as well as a positive infinitesimal ϵ, yields, on combining (2.87, 91, 94),

$$\begin{aligned} V^* = \sum_{i=1}^{m} \left\{\epsilon M_{pi} l_i + \alpha_i (M_{p0} + r_i - M_{pi})\right\} + \sum_{k=1}^{f} \phi_k \left(\sum_{l=1}^{n} F_l U_{lk} - \sum_{j=1}^{h} M_j T_{jk}\right) + \\ + \sum_{j=1}^{h} \left\{\theta'_j \left(M_j + p_j - \sum_{i=1}^{m} \mu_{ji} M_{pi}\right) + \theta''_j \left(-M_j + q_j - \sum_{i=1}^{m} \mu_{ji} M_{pi}\right)\right\}. \end{aligned} \tag{2.95}$$

The conditions for min V^*, subject to the non-negativity conditions of (2.94), can now be written as

$$\left.\begin{aligned} &\sum_{j=1}^{h} (\theta'_j + \theta''_j)\mu_{ji} + \alpha_i = \epsilon l_i \qquad (i = 1, 2 \,..\, m), \\ &\theta_j = \theta'_j - \theta''_j = \sum_{k=1}^{f} T_{jk}\phi_k \qquad (j = 1, 2 \,..\, h), \\ &\theta'_j = 0 \; (p_j > 0), \; \theta'_j \geqslant 0 \; (p_j = 0) \qquad (j = 1, 2 \,..\, h), \\ &\theta''_j = 0 \; (q_j > 0), \; \theta''_j \geqslant 0 \; (q_j = 0) \qquad (j = 1, 2 \,..\, h), \\ &\alpha_i = 0 \; (r_i > 0), \; \alpha_i \geqslant 0 \; (r_i = 0) \qquad (i = 1, 2 \,..\, m), \end{aligned}\right\} \tag{2.96}$$

which by the general theory of Section 1.4 give necessary conditions for min V of (2.87) and together with (2.91, 94) give sufficient conditions for this same minimum.

The Lagrangian multipliers can be given a kinematic interpretation. If $\{\phi_k\}$ are taken as generalized coordinates as in (2.88) defining a virtual deformation (collapse mechanism) of the structure, then $\{\theta_j\}$, as defined in (2.96), gives by (2.88) the corresponding hinge rotations. The quantities $\{\phi_k\}$ and $\{\theta_j\}$ determined by (2.96), do in fact define the collapse mechanism for the failure of the optimum structure. This may be seen by considering the possible forms that the

† Methods for the solution of this problem are given by Heyman (1951), Livesley (1956) and Heyman and Prager (1958).

optimum solution can take for the jth hinge on the ith member ($\mu_{ji} = 1$). If p_j and q_j are both zero, then (2.94) gives a contradiction. The possible cases are thus:

$$\left.\begin{aligned}
&\text{(a)}\ \ p_j = 0,\ q_j > 0,\ r_i \geqslant 0\ \text{ with }\ \theta_j = \theta_j' \geqslant 0,\ \theta_j'' = 0,\ M_j = M_{pi} \geqslant M_{p0},\\
&\text{(b)}\ \ p_j > 0,\ q_j = 0,\ r_i \geqslant 0\ \text{ with }\ \theta_j = -\theta_j'' \leqslant 0,\ \theta_j' = 0,\ M_j = -M_{pi} \leqslant -M_{p0},\\
&\text{(c)}\ \ p_j > 0,\ q_j > 0,\ r_i \geqslant 0\ \text{ with }\ \theta_j = \theta_j' = \theta_j'' = 0,\ |M_j| < M_{pi},\ M_{pi} \geqslant M_{p0},
\end{aligned}\right\} \tag{2.97}$$

where (2.94, 96) have been used. The hinge rotations which occur in the virtual deformation are thus associated with bending moments of the same sign at their limiting plastic values.

It is clear from (2.97) that $|\theta_j| = \theta_j' + \theta_j''$ and so the first of (2.96) may be written as

$$\sum_{\substack{\text{Hinges on}\\ i\text{th member}}} |\theta_j| + \alpha_i = \epsilon l_i, \quad \text{where } \alpha_i \geqslant 0 \qquad (i = 1, 2 \ldots m), \tag{2.98}$$

or recalling (2.94) as

$$\left.\begin{aligned}
&\sum_{\substack{\text{Hinges on}\\ i\text{th member}}} |\theta_j| = \epsilon l_i \qquad (\text{all } i \text{ with } M_{pi} > M_{p0}),\\
&\sum_{\substack{\text{Hinges on}\\ i\text{th member}}} |\theta_j| \leqslant \epsilon l_i \qquad (\text{all } i \text{ with } M_{pi} = M_{p0}),
\end{aligned}\right\} \tag{2.99}$$

which are conditions for the optimum structure, expressed in terms of its collapse mechanism, first given in Foulkes (1954).

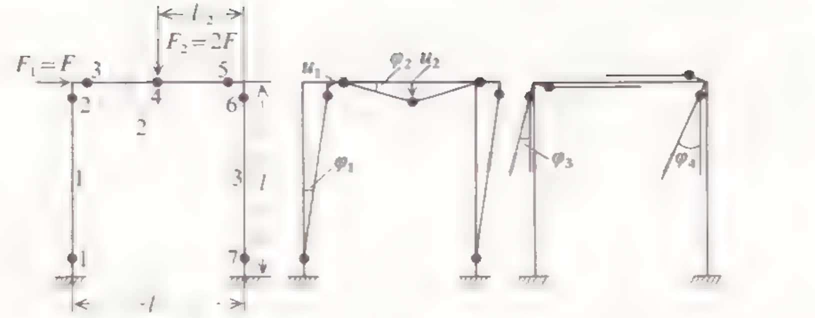
The dual form of the present problem can be obtained by imposing (2.96) on (2.95). This gives, introducing $\{u_l\}$ by (2.88)

$$\max W = (1/\epsilon)\left\{\sum_{l=1}^{n} F_l u_l + M_{p0} \sum_{i=1}^{m} \alpha_i\right\}, \tag{2.100}$$

which is, apart from the term in M_{p0}, the virtual work of the given forces taken over the virtual deformation. The constraints for (2.100) are (2.88, 98). Any solution to the constraints gives a W in (2.100) which is a lower bound to $V_{\min}$. The solution to the dual problem gives $W_{\max} = V_{\min}$ and also determines the active hinges, which by means of (2.91, 97) lead to the optimum design.

2.9 Example of a portal frame

Consider the problem of Fig. 2.2(a) with $M_{p0} = 0$ for simplicity. The possible plastic hinges $j = 1, 2 \ldots 7$ are shown and when they are present the resulting



(a) (b) (c)

Fig. 2.2

mechanism has four degrees of freedom. The deformations corresponding to each of four Lagrangian coordinates $\phi_k (k = 1, 2, 3, 4)$ are shown in (b) and (c). Equations (2.88) now take the form

$$\left.\begin{aligned}
&\theta_1 = \phi_1,\ \theta_2 = \phi_1 - \phi_3,\ \theta_3 = \phi_2 - \phi_3,\ \theta_4 = 2\phi_2,\ \theta_5 = \phi_2 + \phi_4,\\
&\theta_6 = \phi_1 - \phi_4,\ \theta_7 = \phi_1, \quad \text{and} \quad u_1 = l\phi_1,\ u_2 = l\phi_2/2.
\end{aligned}\right\} \tag{2.101}$$

The equations of equilibrium (2.91) can be written down, using the principle of virtual work and (2.101), for each of the deformations $\phi_k (k = 1, 2, 3, 4)$. This gives

$$M_1 + M_2 + M_6 + M_7 = Fl,\ \ M_3 + 2M_4 + M_5 = Fl,\ \ -M_2 - M_3 = 0,\ \ M_5 - M_6 = 0. \tag{2.102}$$

It is to be expected that M_1, M_4, and M_7 will be positive and at their limiting values. Equations (2.102) will allow the same for M_5 and M_6, but not for M_2 and M_3. Equilibrium will allow six bending moments to take the three limiting values for the members and will determine these limiting values and the seventh bending moment. Take as a trial

$$M_1 = M_{p1},\ \ M_3 = M_4 = M_5 = M_{p2},\ \ M_6 = M_7 = M_{p3}. \tag{2.103}$$

Equations (2.102, 103) then give

$$M_{p1} = 3Fl/4,\ \ M_{p2} = M_{p3} = Fl/4,\ \ M_2 = -Fl/4, \tag{2.104}$$

which determines a safe design since $|M_2| < M_{p1}$. Equations (2.97) give $\theta_2 = 0$ and $\theta_i \geqslant 0\ (i = 1, 3 \ldots 7)$. The first of (2.99) and (2.101) give

$$\phi_1 = \epsilon l,\ \ 4\phi_2 - \phi_3 + \phi_4 = \epsilon l,\ \ 2\phi_1 - \phi_4 = \epsilon l \quad \text{and} \quad \phi_1 = \phi_3, \tag{2.105}$$

with a solution,

$$\phi_1 = \phi_3 = \phi_4 = \epsilon l, \quad \phi_2 = \epsilon l/4. \tag{2.106}$$

Equations (2.101, 106) give $\theta_3 < 0$, whereas (2.103, 104) give $M_3 > 0$, contrary to (2.97). The trial of (2.103) has not led to an optimum solution.

The first trial of (2.103) led to a contradiction at the third hinge. This suggests relaxing conditions there and writing

$$M_1 = M_2 = M_{p1}, \quad M_4 = M_5 = M_{p2}, \quad M_6 = M_7 = M_{p3}. \tag{2.107}$$

Equations (2.102, 107) give

$$M_{p1} = Fl/8, \quad M_{p2} = M_{p3} = 3Fl/8, \quad M_3 = -Fl/8, \tag{2.108}$$

with $|M_3| < M_{p2}$. Equations (2.97) give $\theta_3 = 0$ and the rest non-negative. Equations (2.99, 101) give

$$2\phi_1 - \phi_3 = \epsilon l, \quad 3\phi_2 + \phi_4 = \epsilon l, \quad 2\phi_1 - \phi_4 = \epsilon l \quad \text{and} \quad \phi_2 = \phi_3, \tag{2.109}$$

with a solution,

$$\phi_1 = 5\epsilon l/8, \quad \phi_2 = \phi_3 = \phi_4 = \epsilon l/4. \tag{2.110}$$

Equations (2.101, 110) now give $\theta_i > 0$ (all $i, i \neq 3$) and $\theta_3 = 0$, which by (2.97) is consistent with (2.107, 108). The second trial gives an optimum solution. Further confirmation can be obtained by calculating $V_{\min}$ from (2.87, 108) and $W_{\max}$ from (2.100, 101, 110). The result is

$$V_{\min} = W_{\max} = 7Fl^2/8. \tag{2.111}$$

3 Circular sandwich plates

3.1 Plates of least volume of face material

Consider the problem of designing a circular sandwich plate of radius R to balance a system of given normal forces or to transmit such forces to supports at its edge. The sandwich will be assumed to have uniform thickness h, with faces of variable thickness $t(r)$, depending only upon the polar coordinate r (Fig. 3.1). The filling will be assumed to be able to carry such forces as may be required of it.

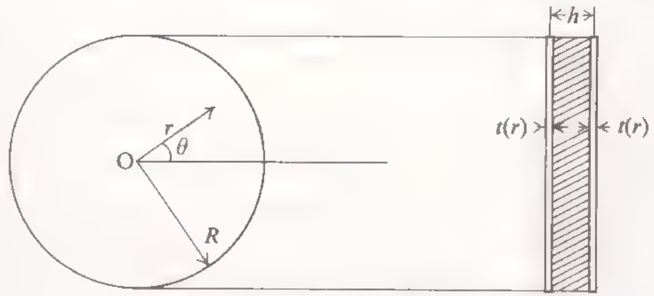


Fig. 3.1

The given forces are assumed to be axially symmetrical. They are defined by the shear force per unit length (stress resultant) $Q(r)$, which gives the total force normal to the plate, within a circle with centre 0 and radius r, as $2\pi r Q$. The function $rQ(r)$ has discontinuities at circles of concentrated forces and tends to a finite limite as $r \to 0$, if there is a concentrated force at the centre.

Let the bending moments per unit length (stress couples) be M_r and M_θ in a polar coordinate system with origin at the centre of the plate. The conditions of equilibrium can then be written as

$$\mathrm{d}(rM_r)/\mathrm{d}r - M_\theta = rQ \tag{3.1}$$

and

$$M_r(r) \text{ continuous}; \quad M_r(R) = 0 \text{ (edge free or simply supported)}. \tag{3.2}$$

The safety of the plate is assured if

$$|M_r|, \ |M_\theta|, \ |M_r - M_\theta| \leq \sigma h t, \tag{3.3}$$

where σ is an allowable stress and the Tresca criterion is assumed to apply to the material of the faces. The relations (3.3) may be written as equations, on introducing slack functions p_i, q_i $(i = 1, 2, 3)$, in the form

$$\left.\begin{aligned} M_r + p_1 &= \sigma ht, & -M_r + q_1 &= \sigma ht,\\ M_\theta + p_2 &= \sigma ht, & -M_\theta + q_2 &= \sigma ht,\\ M_r - M_\theta + p_3 &= \sigma ht, & -M_r + M_\theta + q_3 &= \sigma ht, \end{aligned}\right\} \tag{3.4}$$

where

$$p_1,\ q_1,\ p_2,\ q_2,\ p_3,\ q_3 \geqslant 0. \tag{3.5}$$

Equations (3.3, 4) are given a geometrical representation in Fig. 3.2. The point (M_r, M_θ) must lie within or on the hexagon for safety. The lines upon which the slack functions vanish are shown.

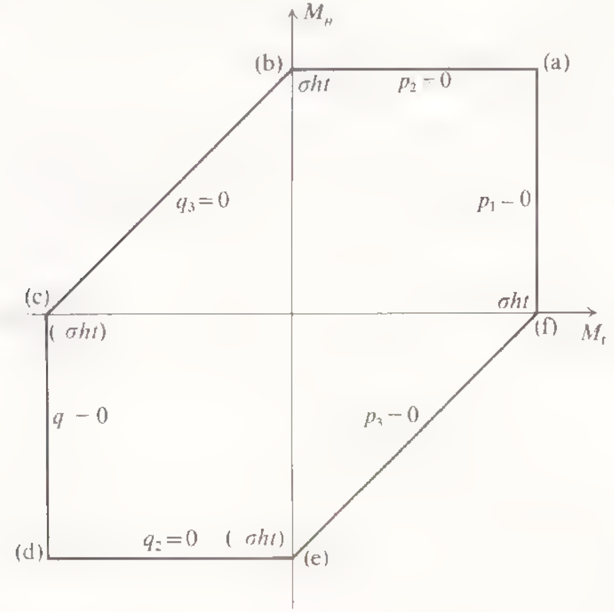


Fig. 3.2

The objective of the present problem is to minimize the material in the face plates. This can be written as

$$\min V = 4\pi \int_0^R tr\,\mathrm{d}r. \tag{3.6}$$

The mathematical problem is thus to minimize V of (3.6) subject to the constraints (3.1, 2, 4, 5).

The problem as formulated is in fact 'irregular' and allows degenerate solutions. To avoid the difficulties that this implies it is necessary to introduce

infinitesimal perturbations of the loading and of the constraints. The addition of a uniformly distributed load $\rho (\rho \to 0)$ over the whole plate adds $\rho r^2/2$ to rQ and enables the assumption to be made that rQ cannot vanish over a finite interval. The second perturbation introduces a factor $1 + \eta\ (\eta \to 0)$ on the right hand side of the second line of (3.4) to replace this by

$$M_\theta + p_2 = \sigma ht(1 + \eta), \quad -M_\theta + q_2 = \sigma ht(1 + \eta). \tag{3.7}$$

Both ρ and η are retained as finite until the analysis of the conditions for a minimum has been completed and are then set equal to zero.

Equations (3.4, 7) or Fig. 3.2 show that it is impossible for three or more slack functions to vanish simultaneously, unless $t = 0$ and all the slack functions vanish. This implies $M_r = M_\theta = 0$ and so by (3.1) $rQ = 0$. It follows that three or more slack functions cannot vanish on an interval. It also follows that t cannot vanish on an interval.

There are six ways in which pairs of slack functions can vanish, without the remainder vanishing. These correspond to the vertices of the hexagon of Fig. 3.2. It is convenient to label points or regions of the plate, to which the conditions at these vertices apply, as follows:

$$\left.\begin{array}{c} \text{(a)}\ p_1 = p_2 = 0, \quad \text{(b)}\ p_2 = q_3 = 0, \quad \text{(c)}\ q_1 = q_3 = 0, \quad \text{(d)}\ q_1 = q_2 = 0,\\ \text{(e)}\ p_3 = q_2 = 0, \quad \text{(f)}\ p_1 = p_3 = 0. \end{array}\right\} \tag{3.8}$$

It will be assumed, similarly to the case of the beam in Section 2.1, that the interval $(0, R)$ can be divided into a finite number of sub-intervals, such that the minimum solution of the present problem satisfies in each of them one of the conditions (a) to (f) of (3.8) or the condition that one or no slack function vanishes in that interval.

Variations from the minimum solution must, by (3.1, 2, 4, 5, 7), satisfy

$$\mathrm{d}(r\,\delta M_r)/\mathrm{d}r - \delta M_\theta = 0, \tag{3.9}$$

$$\delta M_r \text{ continuous}, \quad \delta M_r(R) = 0 \text{ (edge free or simply supported)}, \tag{3.10}$$

$$\left.\begin{aligned} \delta M_r + \delta p_1 &= \sigma h\delta t, & -\delta M_r + \delta q_1 &= \sigma h \delta t,\\ \delta M_\theta + \delta p_2 &= \sigma h\delta t(1 + \eta), & -\delta M_\theta + \delta q_2 &= \sigma h\delta t(1 + \eta),\\ \delta M_r - \delta M_\theta + \delta p_3 &= \sigma h\delta t, & -\delta M_r + \delta M_\theta + \delta q_3 &= \sigma h\delta t, \end{aligned}\right\} \tag{3.11}$$

$$\delta p_i \geqslant 0\ (p_i = 0), \quad \delta q_i \geqslant 0\ (q_i = 0) \quad (i = 1, 2, 3). \tag{3.12}$$

It can now be shown that (3.9, 11) can be solved in ways that enable the following division into dependent and independent variations to be made:

Region	Dependent variations	Independent variations
(a)	$\delta M_r, \delta M_\theta, \delta t, \delta q_1, \delta q_2, \delta p_3, \delta q_3$	$\delta p_1, \delta p_2$
(b)	$\delta M_r, \delta M_\theta, \delta t, \delta p_1, \delta q_1, \delta q_2, \delta p_3$	$\delta p_2, \delta q_3$
(c)	$\delta M_r, \delta M_\theta, \delta t, \delta p_1, \delta p_2, \delta q_2, \delta p_3$	$\delta q_1, \delta q_3$
(d)	$\delta M_r, \delta M_\theta, \delta t, \delta p_1, \delta p_2, \delta p_3, \delta q_3$	$\delta q_1, \delta q_2$
(e)	$\delta M_r, \delta M_\theta, \delta t, \delta p_1, \delta q_1, \delta p_2, \delta q_3$	$\delta p_3, \delta q_2$
(f)	$\delta M_r, \delta M_\theta, \delta t, \delta q_1, \delta p_2, \delta q_2, \delta q_3$	$\delta p_1, \delta p_3$

(3.13)

As an example consider (b). The conditions $p_2 = q_3 = 0$ give, by (3.12), $\delta p_2 \geqslant 0$, $\delta q_3 \geqslant 0$ and these are the only variations that need to be placed in the independent position, since in fact none of the other slack functions or the function t vanish in (b). Equations (3.11) may be solved to give

$$\left.\begin{aligned} \delta p_1 &= -\delta M_r + \sigma h \delta t, & \delta q_1 &= \delta M_r + \sigma h \delta t, \\ \delta M_\theta &= -\delta p_2 + \sigma h \delta t\,(1+\eta), & \delta q_2 &= \delta M_\theta + \sigma h \delta t\,(1+\eta), \\ \delta p_3 &= -\delta M_r + \delta M_\theta + \sigma h \delta t, & \delta M_r &= \sigma h \eta \delta t - \delta p_2 + \delta q_3, \end{aligned}\right\} \tag{3.14}$$

and so, substituting from (3.14) into (3.9), yield

$$\sigma h(\eta r\,\mathrm{d}\delta t/\mathrm{d}r - \delta t) = r\,\mathrm{d}\,\delta p_2/\mathrm{d}r - \mathrm{d}(r\delta q_3)/\mathrm{d}r. \tag{3.15}$$

Equation (3.15) integrates in the form

$$\delta t = \left(C + \int^r [\{r\,\mathrm{d}\delta p_2/\mathrm{d}r - \mathrm{d}(r\delta q_3)/\mathrm{d}r\}/\sigma h \eta r^{1/\eta+1}]\,\mathrm{d}r\right) r^{1/\eta}, \tag{3.16}$$

where C is a constant of integration. Equations (3.14, 16) show that δM_r, δM_θ, δt, δp_1, δq_1, δq_2 and δp_3 can be expressed in terms of δp_2 and δq_3.

The whole interval $(0, R)$† can be split into regions (a) to (f), since for the present purposes a region in which a single slack function vanishes may be included in one of two of the regions (a) to (f) and a region in which no slack function vanishes may be included in any of them. This implies a broadening of the definition of, say, (a) in (3.8) to include regions in which two, one or none of p_1 and p_2 are zero, while the rest of the slack functions are non-zero there. It makes no difference to the conclusions of (3.13). With this interpretation, each interval, into which $(0, R)$ is divided by the minimum solution, has a solution like (3.14, 16) for the variations, and in particular for δM_r, which

† This interval should be taken as 'open' at the end $r = 0$, since solutions with logarithmic infinities at $r = 0$ do occur. All the integrals involved can be shown to be convergent for the actual solutions. These difficulties can be avoided by replacing the lower limit of (3.6) by R_0 and taking the limit $R_0 \to 0$ in the solutions obtained.

contains an arbitrary constant. These constants may be chosen to satisfy the conditions of (3.10), with one to spare when the edge is clamped. It is to be remarked that if η is set equal to zero in (3.15) one of these constants is lost.

The condition for a minimum of V in (3.6) gives

$$\int_0^R \delta t r\,\mathrm{d}r \geqslant 0, \tag{3.17}$$

which on introducing Lagrangian multipliers $r\lambda_i$, $r\mu_i (i = 1, 2, 3)$ and ϕ, as well as the usual positive infinitesimal ϵ, can be written using (3.9, 11) as

$$\begin{aligned} \int_0^R [\sigma h \epsilon r \delta t &+ r\lambda_1(\delta M_r + \delta p_1 - \sigma h \delta t) + r\mu_1(-\delta M_r + \delta q_1 - \sigma h \delta t) + \\ &+ r\lambda_2\{\delta M_\theta + \delta p_2 - \sigma h \delta t(1+\eta)\} + r\mu_2\{-\delta M_\theta + \delta q_2 - \sigma h \delta t(1+\eta)\} + \\ &+ r\lambda_3(\delta M_r - \delta M_\theta + \delta p_3 - \sigma h \delta t) + r\mu_3(-\delta M_r + \delta M_\theta + \delta q_3 - \sigma h \delta t) + \\ &+ \phi\{\mathrm{d}(r\delta M_r)/\mathrm{d}r - \delta M_\theta\}]\,\mathrm{d}r \geqslant 0. \end{aligned} \tag{3.18}$$

Integration of (3.18) by parts, assuming ϕ to be continuous, gives

$$\begin{aligned} \int_0^R \Big[\sigma h\{\epsilon &- \lambda_1 - \mu_1 - (\lambda_2 + \mu_2)(1+\eta) - \lambda_3 - \mu_3\}\delta t + \\ &+ (\lambda_1 - \mu_1 + \lambda_3 - \mu_3 - \mathrm{d}\phi/\mathrm{d}r)\delta M_r + (\lambda_2 - \mu_2 - \lambda_3 + \mu_3 - \phi/r)\delta M_\theta + \\ &+ \sum_{i=1}^{3}(\lambda_i \delta p_i + \mu_i \delta q_i)\Big] r\,\mathrm{d}r + [\phi r \delta M_r]_0^R \geqslant 0. \end{aligned} \tag{3.19}$$

The Lagrangian multipliers must now be chosen so as to remove the dependent variations from (3.19). This requires

$$\left.\begin{aligned} \lambda_1 + \mu_1 + (\lambda_2 + \mu_2)(1+\eta) + \lambda_3 + \mu_3 &= \epsilon, \\ \lambda_1 - \mu_1 + \lambda_3 - \mu_3 &= \mathrm{d}\phi/\mathrm{d}r, \\ \lambda_2 - \mu_2 - \lambda_3 + \mu_3 &= \phi/r, \end{aligned}\right\} \tag{3.20}$$

$$\left.\begin{aligned} &\lambda_3 = \mu_1 = \mu_2 = \mu_3 = 0 \text{ in (a)}, \lambda_1 = \lambda_3 = \mu_1 = \mu_2 = 0 \text{ in (b)}, \lambda_1 = \lambda_2 = \lambda_3 = \mu_2 = 0 \text{ in (c)}, \\ &\lambda_1 = \lambda_2 = \lambda_3 = \mu_3 = 0 \text{ in (d)}, \lambda_1 = \lambda_2 = \mu_1 = \mu_3 = 0 \text{ in (e)}, \lambda_2 = \mu_1 = \mu_2 = \mu_3 = 0 \text{ in (f)}, \end{aligned}\right\} \tag{3.21}$$

and

$$\phi(R) = 0 \text{ (edge } r = R \text{ clamped)}; \quad \phi(0) = 0 \text{ (edge } r = R \text{ free or simply supported)}. \tag{3.22}$$

Equations (3.20, 21) may be solved in each of the regions (a) to (f). For example in region (e), equation (3.21) requires that $\lambda_1 = \lambda_2 = \mu_1 = \mu_3 = 0$ and so from (3.20)

$$\lambda_3 = \mathrm{d}\phi/\mathrm{d}r, \ \mu_2 = -\mathrm{d}\phi/\mathrm{d}r - \phi/r, \ \eta\,\mathrm{d}\phi/\mathrm{d}r + (1+\eta)\phi/r = -\epsilon. \tag{3.23}$$

The last of (3.23) integrates to give

$$\phi = -\epsilon r/(1+2\eta) + K/r^{(1+\eta)/\eta}, \tag{3.24}$$

where K is a constant of integration. The solutions for all the regions contain such constants which may be chosen so that ϕ is continuous, as assumed above, and so that (3.22) is satisfied. It is to be remarked that if $\eta = 0$ the third of (3.23) ceases to be a differential equation and a constant K is lost. The problem is then irregular and the present argument breaks down.

Applying (3.20, 21, 22) to (3.19) gives, using (3.10),

$$\begin{aligned}\int_{(a)} (\lambda_1 \delta p_1 + \lambda_2 \delta p_2) r\,dr + \int_{(b)} (\lambda_2 \delta p_2 + \mu_3 \delta q_3) r\,dr + \int_{(c)} (\mu_1 \delta q_1 + \mu_3 \delta q_3) r\,dr + \\ + \int_{(d)} (\mu_1 \delta q_1 + \mu_2 \delta q_2) r\,dr + \int_{(e)} (\lambda_3 \delta p_3 + \mu_2 \delta q_2) r\,dr + \int_{(f)} (\lambda_1 \delta p_1 + \lambda_3 \delta p_3) r\,dr - \\ - (\phi r \delta M_r)_{r=0} \geqslant 0.\end{aligned} \tag{3.25}$$

It has been assumed in (3.25) that $r\delta M_r$ is finite as $r \to 0$. This can be shown to be true for the solutions of (3.9, 11) in all the regions (a) to (f). The last term in (3.25) is, after (3.22), only required for the clamped edge case and for this, as was remarked above, the solution for δM_r contains an arbitrary constant. This means that the variation δM_r, as $r \to 0$, can be taken as an independent variable in (3.25). The variations of the slack functions in (3.25) are also independent, as is shown by (3.13). Equation (3.25) can thus be dealt with term by term in the usual way, to give, remembering (3.12) and (3.21),

$$\left.\begin{aligned}\lambda_i = 0, \text{ where } p_i > 0 \text{ and } \lambda_i \geqslant 0, \text{ where } p_i = 0 \\ \mu_i = 0, \text{ where } q_i > 0 \text{ and } \mu_i \geqslant 0, \text{ where } q_i = 0\end{aligned}\right\} \quad (i = 1, 2, 3), \tag{3.26}$$

and

$$\phi(0) = 0, \tag{3.27}$$

valid now for all edge conditions.

The necessary conditions for the minimum of (3.6) subject to (3.1, 2, 4, 5) have thus been shown to be the existence of multipliers λ_i, $\mu_i (i = 1, 2, 3)$ and ϕ which satisfy (3.20, 22, 26, 27), with now η written equal to zero.

3.2 The dual problem

The Lagrangian for the problem of (3.1, 2, 4, 5, 6) can be written, using the same multipliers as in (3.18), as

$$\begin{aligned}V^* = (4\pi/\sigma h\epsilon) \int_0^R [\sigma h \epsilon r t + r\lambda_1 (M_r + p_1 - \sigma h t) + r\mu_1(-M_r + q_1 - \sigma h t) + \\ + r\lambda_2 (M_\theta + p_2 - \sigma h t) + r\mu_2(-M_\theta + q_2 - \sigma h t) + r\lambda_3 (M_r - M_\theta + p_3 - \sigma h t) + \\ + r\mu_3(-M_r + M_\theta + q_3 - \sigma h t) + \phi\{ \mathrm{d}(rM_r)/\mathrm{d}r - M_\theta - rQ\}]\,\mathrm{d}r,\end{aligned} \tag{3.28}$$

or on integrating by parts, assuming ϕ to be continuous,†

$$\begin{aligned}V^* = (4\pi/\sigma h\epsilon)\Bigg(\int_0^R \Big\{\sigma h(\epsilon - \lambda_1 - \mu_1 - \lambda_2 - \mu_2 - \lambda_3 - \mu_3) t + \\ + (\lambda_1 - \mu_1 + \lambda_3 - \mu_3 - \mathrm{d}\phi/\mathrm{d}r) M_r + (\lambda_2 - \mu_2 - \lambda_3 + \mu_3 - \phi/r) M_\theta + \\ + \sum_{i=1}^{3} (\lambda_i p_i + \mu_i q_i) - \phi Q\Big\}\, r\,\mathrm{d}r + [\phi r M_r]_0^R\Bigg).\end{aligned} \tag{3.29}$$

The conditions for the minimum of V^* subject to (3.2, 5) and $t \geqslant 0$ are

$$\left.\begin{aligned}&\lambda_1 + \mu_1 + \lambda_2 + \mu_2 + \lambda_3 + \mu_3 = \epsilon \quad (t > 0), \\ &\lambda_1 + \mu_1 + \lambda_2 + \mu_2 + \lambda_3 + \mu_3 \leqslant \epsilon \quad (t = 0), \\ &\lambda_1 - \mu_1 + \lambda_3 - \mu_3 = \mathrm{d}\phi/\mathrm{d}r, \quad \lambda_2 - \mu_2 - \lambda_3 + \mu_3 = \phi/r, \\ &\lambda_i p_i = 0, \quad \lambda_i \geqslant 0, \quad \mu_i q_i = 0, \quad \mu_i \geqslant 0 \quad (i = 1, 2, 3), \\ &\phi(R) = 0 \quad \text{(clamped edge)},\end{aligned}\right\} \tag{3.30}$$

where it has been assumed that $rM_r \to 0$, as $r \to 0$. These agree with those obtained above except the condition for $t = 0$, which was excluded in Section 3.1 and the absence of a condition on $\phi(0)$ from (3.30), except the implied finiteness. It will be seen that in actual solutions multipliers can be found which satisfy the second of (3.30) as an equality ‡ and that (3.30) does in fact imply $\phi(0) = 0$.

The objective for the dual problem, which will be shown to be a maximum problem, is obtained by imposing (3.30) on (3.29). This gives

$$\max W = -(4\pi/\sigma h\epsilon) \int_0^R \phi Q r\,\mathrm{d}r, \tag{3.31}$$

subject to,

$$\left.\begin{aligned}&\lambda_1 + \mu_1 + \lambda_2 + \mu_2 + \lambda_3 + \mu_3 \leqslant \epsilon, \\ &\lambda_1 - \mu_1 + \lambda_3 - \mu_3 = \mathrm{d}\phi/\mathrm{d}r, \\ &\lambda_2 - \mu_2 - \lambda_3 + \mu_3 = \phi/r, \\ &\lambda_i \geqslant 0, \quad \mu_i \geqslant 0 \qquad (i = 1, 2, 3), \\ &\phi(R) = 0 \quad \text{(edge clamped)},\end{aligned}\right\} \tag{3.32}$$

where t and $p_i, q_i (i = 1, 2, 3)$ have been eliminated from (3.30). The dual problem can be given a kinematic interpretation by taking ϕ as the θ-wise virtual component of rotation of the middle surface of the plate. The curvatures

† ϕ continuous is the 'first corner condition' for min V^* (Pars 1962, Section 2.5).

‡ The values of these multipliers are not changed by taking the limit $\rho \to 0$.

are then $d\phi/dr$ and ϕ/r. The objective W is easily seen to be proportional to the virtual work of the external forces and is to be maximized subject to restrictions on the curvatures imposed by (3.32). These restrictions can be shown to be expressible as $|d\phi/dr| + |\phi/r| + |d\phi/dr + \phi/r| \leqslant 2\epsilon$.

The inequality between the W of (3.31), evaluated for a solution of the dual constraints (3.32), and the V of (3.6), evaluated for a solution of what can now be called the primal constraints (3.1, 2, 4, 5), can be established in a similar way to (2.44).

$$
\begin{aligned}
W &= -(4\pi/\sigma h\epsilon)\int_0^R \phi Q r\,dr, \quad \text{(by 3.31)},\\
&= (4\pi/\sigma h\epsilon)\int_0^R \phi\{M_\theta - d(rM_r)/dr\}\,dr, \quad \text{by (3.1)},\\
&= (4\pi/\sigma h\epsilon)\int_0^R \{M_r\,d\phi/dr + M_\theta\phi/r\}\,r\,dr, \quad \text{by (3.2) and last of (3.32)},\\
&= (4\pi/\sigma h\epsilon)\int_0^R \{M_r(\lambda_1-\mu_1+\lambda_3-\mu_3) + M_\theta(\lambda_2-\mu_2-\lambda_3+\mu_3)\}\,r\,dr,\\
&\qquad\qquad \text{by (3.32)},\\
&= (4\pi/\sigma h\epsilon)\int_0^R \left\{\sigma h t\sum_{i=1}^{3}(\lambda_i+\mu_i) - \sum_{i=1}^{3}(\lambda_i p_i + \mu_i q_i)\right\} r\,dr, \quad \text{by (3.4)},\\
&\leqslant 4\pi\int_0^R t r\,dr, \quad \text{by (3.5, 32)},\\
&= V, \quad \text{by (3.6).}
\end{aligned}
\tag{3.33}
$$

The usual deductions can now be made. Equation (3.33) places bounds on $V_{\min}$. The dual problem is a maximum problem. If V has a minimum, then conditions (3.30) are satisfied by the solution and these define a dual solution with $W = V$, which must therefore give both the maximum to W and the minimum to V. The necessary and sufficient conditions that a solution of the primal constraints (3.1, 2, 4, 5) gives min V of (3.6) is that multipliers can be found which satisfy (3.30).

The virtual deformation defined by ϕ may be thought of as a 'collapse mechanism' as in plastic design theory. The argument used to derive (3.33) then shows that the dissipation per unit area of the plate $M_r\,d\phi/dr + M_\theta\phi/r = \sigma h\epsilon t$ for the minimum volume solution given by (3.30). It follows that the dissipation per unit volume is a constant $\sigma h\epsilon/2$. This is a special case of a general theorem given by Drucker and Shield (1956) and derived using the principles of plastic design theory. All the sufficient conditions of the Michell type like (1.70), (2.46), (3.30) and others like them, which have been deduced here using duality theory, are likewise special cases of this general theorem. The application of this theorem to the problems of this chapter is to be found in Onat, Schumann, and Shield (1957). More general loading cases are treated in Prager and Shield (1959).

3.3 Nature of the minimum solution

The minimum solution cannot have a finite interval in which all the slack functions $p_i, q_i (i = 1, 2, 3)$ are non-zero, since this would make, by (3.26), all the multipliers $\lambda_i, \mu_i (i = 1, 2, 3)$ vanish, in contradiction to the first of (3.20).

The minimum solution cannot have a finite interval in which a single slack function vanishes, while the rest are non-zero. This means by (3.26) that all but one of the $\lambda_i, \mu_i (i = 1, 2, 3)$ must be zero. Substitution in (3.20) then leads in all cases to a contradiction. For example, if λ_3 is the only non-zero multiplier, (3.20) gives

$$\lambda_3 = d\phi/dr = -\phi/r = \epsilon, \tag{3.34}$$

which yields inconsistent formulae for ϕ.

The remaining possibilities are the six states (a) to (f) defined by (3.8) and the degenerate state in which all the slack functions and t vanish. All of (a) to (f) are possible on an interval with the exception of (b) and (e). The multipliers for the state (e) are given by (3.23, 24), where the limit $\eta \to 0$ can now be taken. The result is $\phi = -\epsilon r$ and $\lambda_3 = -\epsilon$, which contradicts (3.26). A similar contradiction can be obtained for (b).

The possible solutions valid on finite intervals thus reduce to (a), (c), (d), (f) and the degenerate case. The first four split into two pairs which can be obtained from one another by a reversal of the signs of M_r, M_θ and rQ. The three essentially independent regimes can thus be defined by

$$\text{(A) } p_1 = p_2 = 0, \text{ rest} > 0; \;\; \text{(B) } q_1 = q_3 = 0, \text{ rest} > 0; \;\; \text{(C) } p_i = q_i = 0 \; (i = 1, 2, 3). \tag{3.35}$$

Equations (3.1, 4, 5) give, for intervals $(a, \bar{a})$, $(b, \bar{b})$ and $(c, \bar{c})$ which have regimes (A), (B), and (C) respectively, the results:

$$
\left.
\begin{aligned}
&\text{(A)}\quad M_r = M_\theta = \sigma h t = M_r(\bar{a}) - \int_r^{\bar{a}} Q\,dr > 0 \quad (a < r < \bar{a});\\
&\text{(B)}\quad M_r = -\sigma h t = (b/r)M_r(b) + (1/r)\int_b^r rQ\,dr < 0, \;\; M_\theta = 0 \quad (b < r < \bar{b});\\
&\text{(C)}\quad M_r = M_\theta = t = 0 \quad (c < r < \bar{c}).
\end{aligned}
\right\}
\tag{3.36}
$$

Equations (3.20, 26), with $\eta = 0$, give for the same intervals:

$$
\left.
\begin{aligned}
&\text{(A)}\quad \phi = \epsilon r/2 + A/r, \;\; \lambda_1 = \epsilon/2 - A/r^2, \;\; \lambda_2 = \epsilon/2 + A/r^2, \text{ rest zero } (a < r < \bar{a}),\\
&\qquad \text{with the constant } A \text{ restricted by } -\epsilon a^2/2 \leqslant A \leqslant \epsilon a^2/2;\\
&\text{(B)}\quad \phi = -\epsilon r + B, \;\; \mu_1 = 2\epsilon - B/r, \;\; \mu_3 = -\epsilon + B/r, \text{ rest zero } (b < r < \bar{b}),\\
&\qquad \text{with the constant } B \text{ restricted by } \epsilon\bar{b} \leqslant B \leqslant 2\epsilon b;\\
&\text{(C)}\quad \text{As for (A) or as for (B) } (c < r < \bar{c}).
\end{aligned}
\right\}
\tag{3.37}
$$

The solutions given for (C) are not of course the only ones possible, but are in fact the only ones required.

The regime (B) cannot apply to an interval ending at $r = 0$, since for this $b = 0$ and by (3.37) $\bar{b} = 0$. If the regime (A) applies to an interval ending at $r = 0$, then $a = 0$ and by (3.37) $A = 0$, which removes the singularity at $r = 0$. The only regimes valid near $r = 0$ are (A) and (C) and for both these $\phi(0) = 0$. Finally it is to be remarked that no single regime can give the solution to a clamped edge problem. The only real candidate is (A) and with $A = 0$, (3.37) gives $\phi(R) = \epsilon R/2 \neq 0$.

3.4 Examples

In the case where the edge $r = R$ is simply supported or free, solutions using (A) only are possible. Writing $a = 0, \bar{a} = R$ in (3.36) gives, recalling (3.2),

$$M_r = M_\theta = \sigma ht = -\int_r^R Q\,\mathrm{d}r > 0 \qquad (0 < r < R). \tag{3.38}$$

The conditions of (3.37) impose no restriction in this case. The solution (3.38) is valid so long as $\int_r^R Q\,\mathrm{d}r$ is negative $(0 < r < R)$. If it is positive then the sign of the external loads can be changed. If it changes sign, two or more different regimes must be introduced (see clamped edge solutions below).

The case of uniform loading Z per unit area covering $(R_0 \leqslant r \leqslant R_1)$ gives

$$\left.\begin{aligned} &Q = 0 \qquad (0 \leqslant r \leqslant R_0),\\ &Q = -Z(r^2 - R_0^2)/2r \qquad (R_0 \leqslant r \leqslant R_1),\\ &Q = -Z(R_1^2 - R_0^2)/2r \qquad (R_1 \leqslant r \leqslant R).\end{aligned}\right\} \tag{3.39}$$

Substituting from (3.39) into (3.38) yields

$$\left.\begin{aligned} M_r = M_\theta = \sigma ht &= (Z/2)\{(R_1^2 - R_0^2)/2 - R_0^2 \log(R_1/R_0)\\ &\quad + (R_1^2 - R_0^2)\log(R/R_1)\} \qquad (0 \leqslant r \leqslant R_0),\\ M_r = M_\theta = \sigma ht &= (Z/2)\{(R_1^2 - r^2)/2 - R_0^2 \log(R_1/r)\\ &\quad + (R_1^2 - R_0^2)\log(R/R_1)\} \qquad (R_0 \leqslant r \leqslant R_1),\\ M_r = M_\theta = \sigma ht &= (Z/2)(R_1^2 - R_0^2)\log(R/r) \qquad (R_1 \leqslant r \leqslant R).\end{aligned}\right\} \tag{3.40}$$

The volume is most readily obtained from (3.31), using $\phi = \epsilon r/2$ from (3.37), and (3.39). The result is

$$V = (F/4\sigma h)(2R^2 - R_0^2 - R_1^2), \quad \text{where } F = \pi Z(R_1^2 - R_0^2). \tag{3.41}$$

The special cases of uniform loading over the whole plate $(R_0 = 0, R_1 = R)$ with $V = FR^2/4\sigma h$, concentrated loading at $r = R_0$ $(R_1 = R_0)$ with $V = F(R^2 - R_0^2)/2\sigma h$ and concentrated loading at $r = 0$ $(R_1 = R_0 \to 0)$ with $V = FR^2/2\sigma h$ are readily derived. For this last case

$$M_r = M_\theta = \sigma ht = (F/2\pi)\log(R/r) \qquad (0 < r \leqslant R), \tag{3.42}$$

which gives an unbounded t as $r \to 0$. Note however that $rM_r \to 0$ as assumed above in the derivation of the boundary conditions of (3.30).

The case of a free plate uniformly loaded and supported at the centre by a force F is readily solved as above to give

$$\left.\begin{aligned} &Q = -F/2\pi r + Fr/2\pi R^2,\\ &M_r = M_\theta = \sigma ht = (F/2\pi)\{\log(R/r) - (1 - r^2/R^2)/2\},\\ &V = FR^2/4\sigma h.\end{aligned}\right\} \tag{3.43}$$

In the case when the edge $r = R$ is clamped solutions made up using (A) and (B) are possible. (A) is valid around $r = 0$ and gives, by (3.37), $\phi = \epsilon r/2$. (B) is valid near $r = R$ and so (3.37) and (3.22) give $\phi = \epsilon(R - r)$ and $b \geqslant R/2$. Continuity of ϕ is possible only if $b = 2R/3$, which is $\geqslant R/2$ as required. Continuity of M_r at $r = b = \bar{a}$ requires by (3.36) that t and M_r should vanish at this point.† Equations (3.36) then give

$$\left.\begin{aligned} &M_r = M_\theta = \sigma ht = -\int_r^{2R/3} Q\,\mathrm{d}r > 0 \qquad (0 < r < 2R/3),\\ &M_r = -\sigma ht = (1/r)\int_{2R/3}^r rQ\,\mathrm{d}r < 0, \; M_\theta = 0 \qquad (2R/3 < r < R).\end{aligned}\right\} \tag{3.44}$$

This solution is valid so long as $\int_r^{2R/3} Q\,\mathrm{d}r < 0 \;\; (0 < r < 2R/3)$ and $\int_{2R/3}^r rQ\,\mathrm{d}r < 0 \;\; (2R/3 < r < R)$. If the integrals are positive the sign of the external loads can be changed, but if they have opposite signs or have zeros, more complicated solutions have to be constructed.

The special case for which $\int_r^{2R/3} Q\,\mathrm{d}r \equiv 0$, which occurs when there is no loading inside $r = 2R/3$, can however be dealt with using (C) and (B) only. The regime (C) extends from $r = 0$ to $r = R_1 \geqslant 2R/3$, which is the radius at which loading begins. The multipliers are chosen to be the same as for the previous solution. This means (A) of (3.37) is valid $0 < r < 2R/3$ and (B) of (3.37) for $2R/3 < r < R$. The multipliers for (C) $0 < r < R_1$ are thus chosen as for (A) in part of the range and as for (B) in the rest. This gives a valid solution just as it did previously. The continuity of M_r gives $M_r(R_1) = 0$ in this present case.

† Since M_r is continuous, $t(b - 0) = -t(b + 0)$, with both limits $\geqslant 0$.

Equation (3.36) then gives

$$\left.\begin{aligned} &M_r = M_\theta = t = 0 \quad (0 < r < R_1),\\ &M_r = -\sigma ht = (1/r)\int_{R_1}^{r} rQ\,\mathrm{d}r < 0, \quad M_\theta = 0 \quad (R_1 < r < R),\end{aligned}\right\} \tag{3.45}$$

which is valid when $Q = 0(0 < r < R_1)$, with $R_1 \geqslant 2R/3$, and $\int_{R_1}^{r} rQ\,\mathrm{d}r$ is negative.

The special case of uniform loading for which $Q = -Zr/2$ is solved by (3.44). The results are:

$$\left.\begin{aligned} &M_r = M_\theta = \sigma ht = (Z/4)(4R^2/9 - r^2) \quad (0 < r < 2R/3),\\ &M_r = -\sigma ht = -(Z/6r)(r^3 - 8R^3/27), \; M_\theta = 0 \quad (2R/3 < r < R),\\ &V = 19FR^2/162\sigma h, \;\text{ where } F = \pi R^2 Z.\end{aligned}\right\} \tag{3.46}$$

The case of concentrated loading f per unit length at $r = R_1$, for which $Q = -(fR_1/r)\,1\,(r - R_1)$ is solved by (3.44) or (3.45) according as $R_1 \leqslant 2R/3$ or $\geqslant 2R/3$. The results are:

Case $R_1 \leqslant 2R/3$

$$\left.\begin{aligned} &M_r = M_\theta = \sigma ht = fR_1 \log(2R/3R_1) \quad (0 < r < R_1),\\ &M_r = M_\theta = \sigma ht = fR_1 \log(2R/3r) \quad (R_1 < r < 2R/3),\\ &M_r = -\sigma ht = -(fR_1/r)(r - 2R/3), \; M_\theta = 0 \quad (2R/3 < r < R),\\ &V = (FR^2/3\sigma h)(1 - 3R_1^2/2R^2), \;\text{ where } F = 2\pi R_1 f.\end{aligned}\right\} \tag{3.47}$$

Case $R_1 \geqslant 2R/3$

$$\left.\begin{aligned} &M_r = M_\theta = \sigma ht = 0 \quad (0 < r < R_1),\\ &M_r = -\sigma ht = -(fR_1/r)(r - R_1), \; M_\theta = 0 \quad (R_1 < r < R),\\ &V = (FR^2/\sigma h)(1 - R_1/R)^2, \;\text{ where } F = 2\pi R_1 f.\end{aligned}\right\} \tag{3.48}$$

The case of a concentrated load F at the centre $r = 0$ follows from (3.47), by letting $R_1 \to 0$, with F remaining finite. The result is

$$\left.\begin{aligned} &M_r = M_\theta = \sigma ht = (F/2\pi)\log(2R/3r) \quad (0 < r < 2R/3),\\ &M_r = -\sigma ht = -(F/2\pi r)(r - 2R/3), \; M_\theta = 0 \quad (2R/3 < r < R),\\ &V = FR^2/3\sigma h.\end{aligned}\right\} \tag{3.49}$$

The examples given above are all based upon plastic design. However those for simply supported or free edges, which are based upon solution (A), with the constant $A = 0$, are optimum elastic designs as well. The stresses in the faces are, by (3.38), given by $M_r/ht = M_\theta/ht = \sigma$. The curvatures $\mathrm{d}\phi/\mathrm{d}r$ and ϕ/r are by (3.37) with $A = 0$, given by $\epsilon/2$ in both cases. The corresponding components of strain in the faces are $h\epsilon/4$. The design is elastic, with virtual strains equal to real strains, if $\epsilon = 4\sigma(1 - \nu)/Eh$, where E is Young's modulus and ν is Poisson's ratio.

These elastic designs are uniformly stressed and hence have a uniform density of strain energy. This means that they are also designs of maximum stiffness for given volume of material, since it can be shown that the criterion for this kind of optimum design is that the strain energy is uniformly distributed throughout the structure. The proof of this result for a sandwich plate is effectively the same as that for a plate loaded in its own plane. Reference may therefore be made to Section 5.9 below, where the problem of maximum stiffness for this latter structural form is dealt with.

4 Michell's structural continua

4.1 Michell's sufficient conditions

In Chapter 1 the optimum frameworks were selected from layouts containing a finite number of nodes and members. Complete generality requires that any point in space can be a node and any segment can be occupied by a member. However any attempt to erect a general theory, analogous to that of Section 1.1, using conditions of equilibrium and restrictions on stress levels, is clearly not a practical possibility. Generalization of the dual problem of (1.65, 66) does however lead to new developments, which were first expounded in the important paper Michell (1904).

The problem of (1.65, 66) can be formulated in completely general terms as follows:

Find that virtual displacement of space, subject to the kinematic conditions imposed on the frameworks and to the restriction that the overall strain for all segments lies between $-\sigma\epsilon/\sigma_C$ and $\sigma\epsilon/\sigma_T$, which is such that the virtual work of the given external forces is a maximum. (4.1)

This is a problem of the calculus of variations and, as in all such problems, progress can only be made if the mathematical character of the unknown functions is restricted (Pars 1962, Section 1.3). Let is be assumed that the virtual strain of (4.1) is piece-wise continuous. This means that the Cartesian components of virtual displacement are continuously differentiable functions† of the Cartesian coordinates, except at a finite number of surfaces of discontinuity. This is the usual assumption made in the derivation of a theory of strain and leads in the well known way to the existence of principal strains and principal directions. The statement of (4.1) can now be altered to read:

Find that virtual displacement of space, subject to the kinematic conditions imposed on the frameworks and to the restriction that the principal strains everywhere lie between $-\sigma\epsilon/\sigma_C$ and $\sigma\epsilon/\sigma_T$, which is such that the virtual work of the given external forces is a maximum. (4.2)

This is valid since the condition of (4.1) implies that all direct strain components at all points of space lie between the limits $-\sigma\epsilon/\sigma_C$ and $\sigma\epsilon/\sigma_T$ and since, conversely, the condition of (4.2) implies that the overall strain for any segment lies between these same limits.

† Up to the third order at least, if a theory of compatibility of strain is to be developed.

If (4.2) has a solution, then the minimum volume required for a framework, which carries the given forces, may be expected to be given by the maximum virtual work divided by $\sigma\epsilon$. The layout of the corresponding structure may be expected to be given by those segments, which have strains equal to $-\sigma\epsilon/\sigma_C$ or $\sigma\epsilon/\sigma_T$ and define possible compression or tension members. This gives definite locations for members when the lines of principal strain are straight lines. In general however such lines are curved and so possible members can only lie along those infinitesimal segments into which the lines of principal strain may be imagined to be divided. The attempt to generalize the theory of Chapter 1, to allow for all possible layouts, has thus led to a theory of frameworks with curvilinear members, which, as equilibrium of individual members shows, must form a continuum.

The transition from (4.1) to (4.2) requires the introduction of special assumptions and so, even if (4.1) generates all optimum frameworks, the formulation of (4.2) may well only generate a sub-class. Sufficient conditions, corresponding to (4.2), that a framework should be an optimum, are easily established. There is, however, no assurance that such a structure will exist, for all possible systems of given forces. A large number of optimum structures have been found, nevertheless, by the use of these conditions and so their study is an important part of the subject of optimum frameworks.

Michell's sufficient conditions, for a framework to have the least volume of material, can be formulated as follows:

A pin-jointed framework has the least volume of material, if it can carry its given forces, with stresses in its tension members equal to σ_T and stresses in its compression members equal to $-\sigma_C$ and if a virtual deformation of a region of space, in which the competing frameworks must lie, satisfies the kinematic conditions imposed on the framework and gives strains of $\sigma\epsilon/\sigma_T$ in its tension members, strains of $-\sigma\epsilon/\sigma_C$ in its compression members and has no direct strain lying outside these limits. (4.3)

The formulation of (4.3) restricts the frameworks to a definite region of space. This is normally the case in practice, but the whole of space may be used if 'absolute optima' are required. The theorem of (4.3) is a generalization of (1.70) at least as far as sufficient conditions are concerned. Structures which satisfy the conditions of (4.3) or similar conditions may be called 'Michell structures'.

A proof of (4.3) may be given for any kind of framework with discrete and/or continuous distributions of members. The class of all competing frameworks, which can safely carry the given forces and which includes the Michell framework, is subjected to the virtual deformation of (4.3). The virtual work W is the same for all the frameworks, since the forces and deformations are the same. The principle of virtual work gives

$$W = \sum T\gamma l, \tag{4.4}$$

where T is a typical end load in a member, γ the corresponding virtual strain

and l the length of the member. The 'sum' $\sum$ sums over all members of a framework and must be replaced by appropriate integrals for continuous distributions of members.

For the Michell framework

$$\left.\begin{aligned} T\gamma &= (\sigma_T A)(\sigma\epsilon/\sigma_T) = \sigma\epsilon A \quad &\text{(tension members)},\\ T\gamma &= (-\sigma_C A)(-\sigma\epsilon/\sigma_C) = \sigma\epsilon A \quad &\text{(compression members)},\end{aligned}\right\} \tag{4.5}$$

where A is the area of cross-section of the corresponding member. Equations (4.4, 5) give

$$W = \sigma\epsilon \sum Al = \sigma\epsilon V_m \tag{4.6}$$

where V_m is the volume of the Michell framework.

For any of the frameworks under consideration

$$\left.\begin{aligned} &T\gamma \leqslant T(\sigma\epsilon/\sigma_T) \leqslant \sigma\epsilon A \quad &\text{(tension members)},\\ &T\gamma = (-T)(-\gamma) \leqslant (-T)(\sigma\epsilon/\sigma_C) \leqslant \sigma\epsilon A \quad &\text{(compression members)},\end{aligned}\right\} \tag{4.7}$$

since $-\sigma_C A \leqslant T \leqslant \sigma_T A$ and $-\sigma\epsilon/\sigma_C \leqslant \gamma \leqslant \sigma\epsilon/\sigma_T$. Equations (4.4, 7) give

$$W \leqslant \sigma\epsilon \sum Al = \sigma\epsilon V, \tag{4.8}$$

where V is the volume of any framework which carries the given forces.

Equations (4.6, 8) complete the proof by giving

$$V_m \leqslant V. \tag{4.9}$$

A special case of (4.3) arises when a framework, whose given supports, if any, are not redundant, carries the given forces with all its members in tension with stress σ_T or in compression with stress $-\sigma_C$. Such a framework is an optimum, since it allows a uniform dilatation of space with linear strain ϵ or $-\epsilon$, as the case may be. This result was deduced by Maxwell from a more general theorem on frameworks which are stressed to their limits (Maxwell, 1890). This theorem can be stated in the form

$$\sigma_T V_T - \sigma_C V_C = W_0/\epsilon, \tag{4.10}$$

where V_T is the volume of tension members, V_C the volume of compression members and W_0 the virtual work of the external forces for a uniform dilatation with linear strain ϵ. The proof of (4.10) follows immediately from (4.4). If V_C (or V_T) is absent from (4.10), it follows that all frameworks, which carry given forces with their members fully stressed in tension (or compression), have equal volumes of material. Many interesting examples of single-stress structures and of the application of Maxwell's theorem (4.10) are to be found in Cox (1965).

A number of general conclusions can be drawn about the layouts of Michell frameworks and their associated strain fields from the properties of principal strains and directions. If a single tension (compression) member passes through a point then its direction has a principal strain $\sigma\epsilon/\sigma_T$ $(-\sigma\epsilon/\sigma_C)$ and the principal strains at right angles are limited in magnitude. If a pair of distinct tension (compression) members passes through a point then the direct strains in all directions in their plane are $\sigma\epsilon/\sigma_T$ $(-\sigma\epsilon/\sigma_C)$ and any number of coplanar tension (compression) members are possible; the principal strain at right angles to these members is limited in magnitude. If a pair of tension and compression members meet at a point, they must be orthogonal and have strains $\sigma\epsilon/\sigma_T$ and $-\sigma\epsilon/\sigma_C$. No other members can be coplanar with them and the principal strain at right angles must be limited in magnitude. If three non-coplanar tension (compression) members meet at a point, the direct strain in any direction at this point must be $\sigma\epsilon/\sigma_T(-\sigma\epsilon/\sigma_C)$. If two tension (compression) members and one compression (tension) member meet at a point, then the compression (tension) member must be at right angles to the other two, the strain in any direction in the plane of the tension (compression) members must be $\sigma\epsilon/\sigma_T$ $(-\sigma\epsilon/\sigma_C)$ and the strain in the direction of the compression (tension) member must be $-\sigma\epsilon/\sigma_C\,(\sigma\epsilon/\sigma_T)$.

Michell layouts in three-dimensions have a very special character. They have been little studied however. Most progress has been made in two-dimensions, with which this chapter is mainly concerned.

4.2 Two-dimensional strain fields

The case where the two principal strains are both $\sigma\epsilon/\sigma_T$ or $-\sigma\epsilon/\sigma_C$ gives no guidance on layout at all. All members are in tension or alternatively in compression and can be arranged in any way that is suitable for carrying the given forces. As was seen above, the volume of material required is independent of the layout.

The case where one principal strain is $\sigma\epsilon/\sigma_T$ and the other $-\sigma\epsilon/\sigma_C$ gives orthogonal layouts of members, which provide most of the known Michell frameworks. The analysis of this strain field is the immediate task of this section.

A system of strain with principal strains $\sigma\epsilon/\sigma_T$ and $-\sigma\epsilon/\sigma_C$ will have a definite orthogonal system of lines of principal strain. These may be used to define a curvilinear coordinate system with coordinates (α, β) (Fig. 4.1). This system will be taken as right-handed† with strains $\sigma\epsilon/\sigma_T$ in the direction of the α-lines and strains $-\sigma\epsilon/\sigma_C$ in the direction of the β-lines. The line element ds will be given by an expression of the form

$$ds^2 = A^2\,d\alpha^2 + B^2\,d\beta^2, \tag{4.11}$$

where A and B are positive functions of α and β. If ϕ is the angle between the axis $0x$ of a Cartesian coordinate system and the tangent to an α-line in the direction of α increasing, then

$$\partial x/A\,\partial\alpha = \cos\phi, \quad \partial y/A\,\partial\alpha = \sin\phi, \quad \partial x/B\,\partial\beta = -\sin\phi, \quad \partial y/B\,\partial\beta = \cos\phi. \tag{4.12}$$

† A rotation from the positive direction on an α-line to the positive direction on a β-line agrees in sense with that from $0x$ to $0y$.

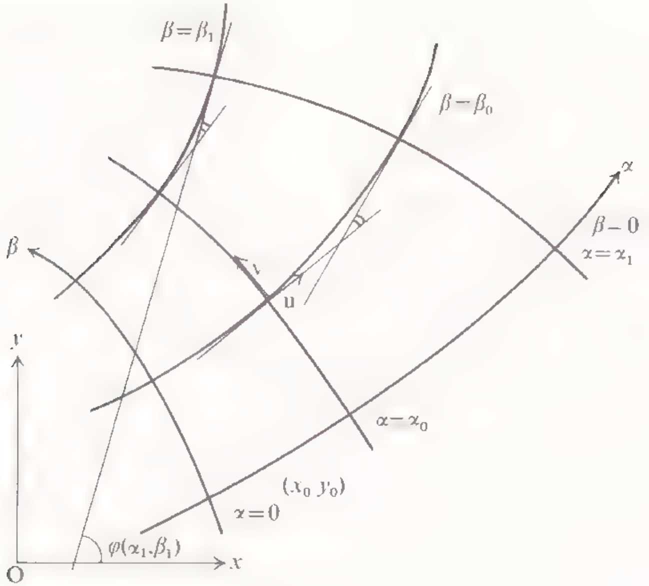


Fig. 4.1

Integration of (4.12) yields

$$x + iy = x_0 + iy_0 + \int_{(0,\,0)}^{(\alpha,\,\beta)} e^{i\phi}(A\,d\alpha + iB\,d\beta), \tag{4.13}$$

where (x_0, y_0) is the point at which $\alpha = \beta = 0$. The condition of single-valuedness of (4.13) gives

$$\partial\phi/\partial\alpha = -\partial A/B\,d\beta, \quad \partial\phi/\partial\beta = \partial B/A\,\partial\alpha, \tag{4.14}$$

which integrates in the form

$$\phi = \phi_0 + \int_{(0,\,0)}^{(\alpha,\,\beta)} \left\{(-\partial A/B\partial\beta)\,d\alpha + (\partial B/A\partial\alpha)\,d\beta\right\}, \tag{4.15}$$

where ϕ_0 is the value of ϕ when $\alpha = \beta = 0$. The single-valuedness of (4.15) gives

$$(\partial/\partial\alpha)(\partial B/A\partial\alpha) + (\partial/\partial\beta)(\partial A/B\partial\beta) = 0, \tag{4.16}$$

which all such functions A, B must satisfy.

Let the displacement components along the coordinate lines in the directions of α and β increasing be denoted by u and v. The direct strains in the α and β directions are $\sigma\epsilon/\sigma_T$ and $-\sigma\epsilon/\sigma_C$ respectively and the associated shear strain is zero. Denoting the rotation by ω, the standard formulae in curvilinear coordinates† give

$$\left.\begin{aligned} \partial u/A\partial\alpha - v\partial\phi/A\partial\alpha = \sigma\epsilon/\sigma_T, \quad \partial v/B\partial\beta + u\partial\phi/B\partial\beta = -\sigma\epsilon/\sigma_C, \\ \partial v/A\partial\alpha + \partial u/B\partial\beta + u\partial\phi/A\partial\alpha - v\partial\phi/B\partial\beta = 0, \\ \partial v/A\partial\alpha - \partial u/B\partial\beta + u\partial\phi/A\partial\alpha + v\partial\phi/B\,d\beta = 2\omega. \end{aligned}\right\} \tag{4.17}$$

Solution of (4.17) for the derivatives of u and v yields

$$d\{e^{i\phi}(u + iv)\} = e^{i\phi}\{\sigma\epsilon(A\,d\alpha/\sigma_T - iB\,d\beta/\sigma_C) + i\omega(A\,d\alpha + iB\,d\beta)\}, \tag{4.18}$$

which integrates to give

$$u + iv = e^{i(\phi_0 - \phi)}(u_0 + iv_0) + e^{-i\phi}\int_{(0,\,0)}^{(\alpha,\,\beta)} e^{i\phi}\{\sigma\epsilon(A\,d\alpha/\sigma_T - iB\,d\beta/\sigma_C) + i\omega(A\,d\alpha + iB\,d\beta)\}, \tag{4.19}$$

where u_0, v_0 are the displacements at $\alpha = \beta = 0$. The single-valuedness of (4.19) requires that

$$\partial\omega/\partial\alpha = \sigma\epsilon(1/\sigma_T + 1/\sigma_C)\partial\phi/\partial\alpha, \quad \partial\omega/\partial\beta = -\sigma\epsilon(1/\sigma_T + 1/\sigma_C)\partial\phi/\partial\beta, \tag{4.20}$$

which integrates to give

$$\omega = \omega_0 + \sigma\epsilon(1/\sigma_T + 1/\sigma_C)\int_{(0,\,0)}^{(\alpha,\,\beta)} \{(\partial\phi/\partial\alpha)\,d\alpha - (\partial\phi/\partial\beta)\,d\beta\}, \tag{4.21}$$

where ω_0 is the value of ω, when $\alpha = \beta = 0$. The single-valuedness of (4.21) gives

$$\partial^2\phi/\partial\alpha\partial\beta = 0, \tag{4.22}$$

which characterizes the layout of Michell frameworks. The result of (4.22) can also be written, using (4.14), as

$$(\partial/\partial\alpha)(\partial B/A\partial\alpha) = (\partial/\partial\beta)(\partial A/B\partial\beta) = 0, \tag{4.23}$$

which are equations for A and B. Equation (4.22) can be integrated over a curvilinear rectangle bounded by $\alpha = \alpha_0$, $\alpha = \alpha_1$, $\beta = \beta_0$ and $\beta = \beta_1$ to give

$$\phi(\alpha_1, \beta_0) - \phi(\alpha_0, \beta_0) = \phi(\alpha_1, \beta_1) - \phi(\alpha_0, \beta_1). \tag{4.24}$$

This shows that the angle turned through by the tangent to an α-line ($\beta = \beta_0$), as it moves between two fixed β-lines ($\alpha = \alpha_0, \alpha_1$), is equal to the angle obtained by using the α-line ($\beta = \beta_1$) instead or indeed any other α-line. The angles involved here are marked on Fig. 4.1. The same result is found if the roles of the α and β-lines are interchanged. This geometrical property is the characteristic of sets of orthogonal curves known as Hencky nets. Slip lines in two-dimensional

† See, e.g. Love (1927), Chap. 1, equations 36 and 38.

perfect plastic flow form such nets. A knowledge of slip-line fields is thus of great value for the discovery of Michell layouts and the techniques for calculating the form of slip lines can be used in this new application. Reference may be made to Hill (1950) Chap. VI and to Geiringer (1937) for an account of these techniques.†

The boundary conditions at a fixed line of support, along which $u + iv = 0$, may also be noted. Equation (4.18) gives

$$\omega = \pm\sigma\epsilon/\sqrt{(\sigma_T\sigma_C)}, \quad A\,d\alpha = \pm\sqrt{(\sigma_T/\sigma_C)}B\,d\beta. \tag{4.25}$$

The first of (4.25) determines ω on the fixed line, while the second gives the angle between the α or β-lines and the tangent to that line.

States of strain in which only one of the principal strains is at its limit are also possible and give layouts consisting of a single set of non-intersecting members. These must all be straight lines, unless suitable distributed forces act in the region concerned. Consideration will therefore be limited here to a system of straight lines, which are assumed to have strains $\sigma\epsilon/\sigma_T$ and therefore to be α-lines. These lines will envelop an evolute and the β-lines will be involutes (Fig. 4.2). The coordinate β can be taken equal to ϕ and the α-coordinate can be taken as the distance along an α-line measured from a fixed involute. It follows that

$$A = 1, \quad B = \alpha + F(\beta), \tag{4.26}$$

where $F(\beta)$ is an arbitrary function depending on the form of the evolute, which will have the equation $\alpha + F(\beta) = 0$. Finally

$$\phi = \beta, \quad x + iy = x_0 + iy_0 + \alpha e^{i\beta} + i\int_0^\beta e^{i\beta}F(\beta)d\beta, \tag{4.27}$$

where the second result follows from (4.13, 26).

The strain in the α-lines is $\sigma\epsilon/\sigma_T$. Let the strain in the β-lines be ϵ_2. Then equations like (4.17) can be written, taking into account the special forms of (4.26, 27). This gives

$$\left.\begin{aligned} &\partial u/\partial\alpha = \sigma\epsilon/\sigma_T, \quad (\partial v/\partial\beta + u)/\{\alpha + F(\beta)\} = \epsilon_2,\\ &\partial v/\partial\alpha + (\partial u/\partial\beta - v)/\{\alpha + F(\beta)\} = 0,\\ &\partial v/\partial\alpha - (\partial u/\partial\beta - v)/\{\alpha + F(\beta)\} = 2\omega. \end{aligned}\right\} \tag{4.28}$$

The first and third of (4.28) integrate to give

$$u = \sigma\epsilon\alpha/\sigma_T + G(\beta), \quad v = G'(\beta) + \{\alpha + F(\beta)\}\,H(\beta), \tag{4.29}$$

† This analogy with plasticity theory was noted by Prager (1958) and by Hemp (1958). Prager also gives a powerful method for the graphical construction of Hencky nets.

where G, H are arbitrary functions. Equation (4.28) then gives for ω and ϵ_2 the results:

$$\left.\begin{aligned} &\omega = H(\beta),\\ &\epsilon_2 = H'(\beta) + \{\epsilon\alpha + G''(\beta) + F'(\beta)\,H(\beta) + G(\beta)\}\,/\,\{\alpha + F(\beta)\}. \end{aligned}\right\} \tag{4.30}$$

This system of strain is valid for a Michell structural layout if

$$-\sigma\epsilon/\sigma_C \leqslant \epsilon_2 \leqslant \sigma\epsilon/\sigma_T. \tag{4.31}$$

4.3 Conditions of equilibrium

The structure corresponding to a strain field with principal strains $\sigma\epsilon/\sigma_T$, $-\sigma\epsilon/\sigma_C$ will have, say, members along α-lines with 'equivalent thickness' or cross-sectional area per unit width t_1, and members along β-lines with equivalent thickness t_2. The end load carried in the α-direction by an element with coordinate difference $d\beta$ will be $\sigma_T t_1 B\,d\beta$ and that carried in the β-direction by an element with coordinate difference $d\alpha$ will be $-\sigma_C t_2 A\,d\alpha$. It is convenient to introduce

$$\mathbf{T}_1 = \sigma_T B t_1, \quad \mathbf{T}_2 = -\sigma_C A t_2, \tag{4.32}$$

which are end loads per unit coordinate difference in the α and β directions respectively.

The differential equations of equilibrium can be written down using standard results for curvilinear coordinates.† These give

$$\partial\mathbf{T}_1/\partial\alpha - \mathbf{T}_2\,\partial\phi/\partial\beta = 0, \quad \partial\mathbf{T}_2/\partial\beta + \mathbf{T}_1\,\partial\phi/\partial\alpha = 0, \tag{4.33}$$

in the absence of forces in the body of the structure.

Boundary conditions at edges, which may carry 'concentrated' members of finite cross-sectional area, are easily established. For an edge lying along a β-line, with the structure on the side of α decreasing, which has a concentrated member carrying a compressive load P and a distribution of external forces of magnitudes F_α, F_β per unit length in the α and β directions respectively, it follows that for equilibrium

$$\mathbf{T}_1 = P\partial\phi/\partial\beta + F_\alpha B, \quad \partial P/\partial\beta = F_\beta B. \tag{4.34}$$

The corresponding result for an edge along an α-line, with a member carrying a tension T, is

$$\mathbf{T}_2 = T\partial\phi/\partial\alpha + F_\beta A, \quad \partial T/\partial\alpha = -F_\alpha A. \tag{4.35}$$

The values of T and P follow from the second of equations (4.34, 35), together with initial values of end load induced by a given concentrated force, which requires concentrated members for its transmission.

† See, e.g. Love (1927), Section 331. However (4.33) are easily established directly.

The sizes of the distributed members follow from (4.32). The 'total equivalent thickness' t is given by

$$t = t_1 + t_2 = \mathbf{T}_1/\sigma_T B - \mathbf{T}_2/\sigma_C A. \tag{4.36}$$

The structure corresponding to a straight line system with strain $\sigma\epsilon/\sigma_T$ and with the second principal strain not at either limit, will have $t_2 = 0$. Equation (4.33) gives $\mathbf{T}_1$ as a function of β only, as it should, and so in this case $\mathbf{T}_1$ follows from a boundary condition of the form of (4.34). Equations (4.26, 32) give

$$t_1 = \mathbf{T}_1(\beta)/\sigma_T\{\alpha + F(\beta)\}. \tag{4.37}$$

4.4 Mathematical considerations

The simplest Hencky net consists of two sets of parallel straight lines which are orthogonal to each other. For these α, β can be taken as Cartesian coordinates with $A = B = 1$. The angle ϕ is constant and (4.13) gives the usual relation between two distinct Cartesian coordinate systems. Equation (4.21) gives ω = constant and (4.19) gives

$$u = u_0 - \omega_0\beta + \sigma\epsilon\alpha/\sigma_T, \quad v = v_0 + \omega_0\alpha - \sigma\epsilon\beta/\sigma_C, \tag{4.38}$$

which is a rigid body motion combined with strains $\sigma\epsilon/\sigma_T, -\sigma\epsilon/\sigma_C$ in the coordinate directions.

If one coordinate line, say an α-line, is straight, then, by (4.22) or (4.24), it follows that all the α-lines are straight. These α-lines will in general envelop an evolute as in Fig. 4.2. Equations (4.26, 27) are then applicable. Equation (4.21)

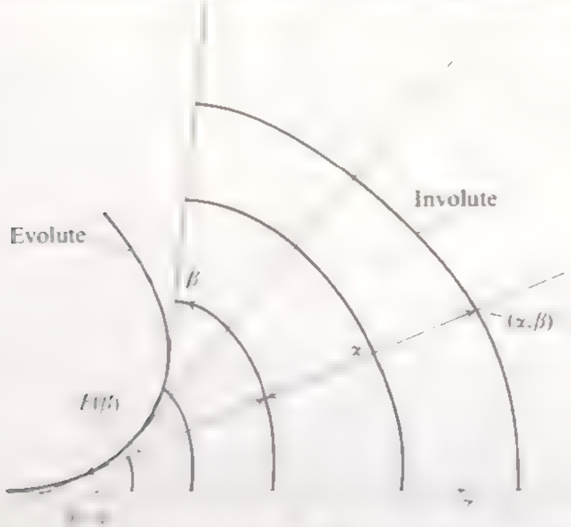


Fig. 4.2

gives

$$\omega = \omega_0 - \sigma\epsilon\beta(1/\sigma_T + 1/\sigma_C), \tag{4.39}$$

and equation (4.19) yields

$$u + iv = (u_0 + iv_0)e^{-i\beta} + [\sigma\epsilon/\sigma_T + i\{\omega_0 - \sigma\epsilon\beta(1/\sigma_T + 1/\sigma_C)\}]\alpha + $$
$$+ e^{-i\beta}\int_0^\beta \{-i\sigma\epsilon/\sigma_C - \omega_0 + \sigma\epsilon\beta(1/\sigma_T + 1/\sigma_C)\}Fe^{i\beta}\,d\beta. \tag{4.40}$$

Equations (4.33) integrate in the form

$$\mathbf{T}_1 = T_{10}(\beta) + \int_0^\alpha T_2(\alpha)\,d\alpha, \quad \mathbf{T}_2 = T_2(\alpha), \tag{4.41}$$

where the functions $T_{10}(\beta)$ and $T_2(\alpha)$ follow from (4.34, 35).

In the case where both coordinate lines are curved it is possible to define the coordinates so that

$$\phi = \phi_0 - a\alpha + b\beta, \quad a, b = +1 \text{ or } -1. \tag{4.42}$$

This requires $\alpha = -a(\phi - \phi_0)$ on $\beta = 0$ and $\beta = b(\phi - \phi_0)$ on $\alpha = 0$. It is thus necessary to define α as the angle through which the tangent to $\beta = 0$ turns, as it moves from $\alpha = 0$ in the direction of α increasing, and to define β as the corresponding angle on $\alpha = 0$ (Fig. 4.3). The choice of the signs of a and b depends upon the direction of rotation of these tangents. In Fig. 4.3, $a = b = 1$,

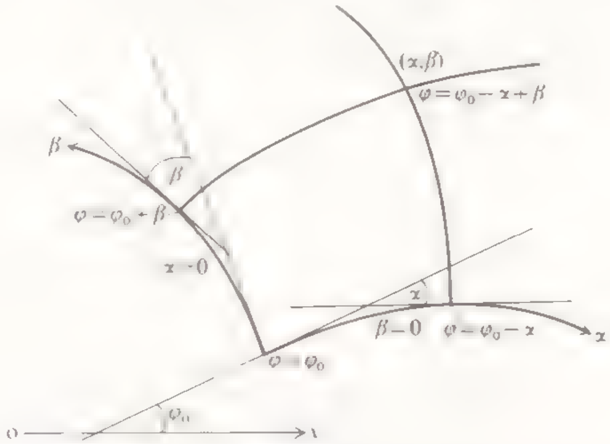


Fig. 4.3

but three other combinations of sign will be obtained if the curvature of one or both of the lines $\beta = 0$, $\alpha = 0$ is reversed. Equation (4.42) is a consequence of these definitions and the rule of (4.24). It is clear that this coordinate system will only be valid between points of inflexion on $\beta = 0$ and $\alpha = 0$.

Equations (4.14) now give

$$\partial A/\partial\beta = aB, \quad \partial B/\partial\alpha = bA, \tag{4.43}$$

and so A, B satisfy

$$\partial^2 A/\partial\alpha\partial\beta = cA, \quad \partial^2 B/\partial\alpha\partial\beta = cB, \quad c = ab. \tag{4.44}$$

Equations (4.44) can be integrated by Riemann's method. A particular solution is given by

$$A = I_0(t), \quad t = 2\sqrt{\{c(\alpha-\xi)(\beta-\eta)\}}, \tag{4.45}$$

where I_0 is the modified Bessel function, which satisfies

$$d^2I_0/dt^2 + dI_0/t\,dt - I_0 = 0, \quad I_0(0) = 1, \tag{4.46}$$

and ξ, η are parameters. It follows that

$$\begin{aligned}(\partial/\partial\alpha)(A\,\partial I_0/\partial\beta) - (\partial/\partial\beta)(I_0\,\partial A/\partial\alpha) = A\,(\partial^2 I_0/\partial\alpha\partial\beta - cI_0) - \\ - I_0(\partial^2 A/\partial\alpha\partial\beta - cA) = 0,\end{aligned} \tag{4.47}$$

and so by Green's theorem that

$$\oint\{(I_0\,\partial A/\partial\alpha)\,d\alpha + (A\,\partial I_0/\partial\beta)\,d\beta\} = 0. \tag{4.48}$$

Application of (4.48) to the circuit DO_1CPD of Fig. 4.4, where CP is a

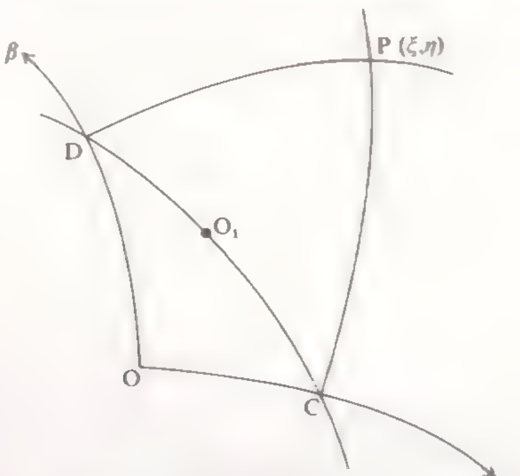


Fig. 4.4

β-line, DP an α-line and P the point (ξ, η), gives, since $(I_0)_{\alpha=\xi} = (I_0)_{\beta=\eta} = 1$,

$$A(\xi, \eta) = A_D + \int_{DO_1C} \{(I_0\,\partial A/\partial\alpha)\,d\alpha + (A\,\partial I_0/\partial\beta)\,d\beta\} \tag{4.49}.$$

Equation (4.49) gives A at P, when A and $\partial A/\partial\alpha$ are known on DO_1C. If DO_1C is replaced by DOC, where OC is an α-line and OD a β-line, then (4.49) yields, after an integration by parts,

$$A(\xi, \eta) = A_0 I_0\{2\sqrt{(c\xi\eta)}\} + \int_{OC} (I_0\,\partial A/\partial\alpha)\,d\alpha + \int_{OD} (I_0\,\partial A/\partial\beta)\,d\beta. \tag{4.50}$$

Equation (4.50) gives A at P when $\partial A/\partial\alpha$ is known on OC and $\partial A/\partial\beta = aB$ is known on OD.

A formula for B can now be obtained from the first of (4.43). Equation (4.13) can be used to derive x and y. Equation (4.21) gives, using (4.42),

$$\omega = \omega_0 - \sigma\epsilon(1/\sigma_T + 1/\sigma_C)(a\alpha + b\beta), \tag{4.51}$$

and (4.19) can be integrated to give u and v.

Equations (4.33), (4.42) give

$$\partial T_2/\partial\beta = aT_1, \quad \partial T_1/\partial\alpha = bT_2, \tag{4.52}$$

which can be integrated to give T_1 and T_2 in the same way as A and B follow from (4.43).

4.5 Straight lines and circles

A particular case of the Hencky net, which consists of straight lines and involutes, arises when the evolute contracts to a point. This gives a set of concurrent straight lines and an orthogonal set of concentric circles. Such a figure clearly satisfies (4.24). It is convenient to use polar coordinates and to write

$$\alpha = r, \quad \beta = \theta. \tag{4.53}$$

Equation (4.27) and (4.26), with $F(\beta) = 0$, give

$$\phi = \theta, \quad A = 1, \quad B = r. \tag{4.54}$$

Equations (4.39, 40), with $\omega_0 = u_0 = v_0 = 0$, give, on changing the notation for the displacements,

$$\omega = -\sigma\epsilon(1/\sigma_T + 1/\sigma_C)\theta, \quad u_r = \sigma\epsilon r/\sigma_T, \quad u_\theta = -\sigma\epsilon(1/\sigma_T + 1/\sigma_C)r\theta, \tag{4.55}$$

where it is to be remarked that ω and u_θ are multivalued and so the virtual deformation of (4.55) cannot be applied to the whole plane.

This layout of circles and radii can be used to solve the problem shown in Fig. 4.5.† Here an optimum structure is sought to balance the forces shown and to lie inside one of the half-planes defined by the line through their points of application. The solution consists of a concentrated member of area F/σ_C at

† All the problems of this section were solved in Michell (1904).

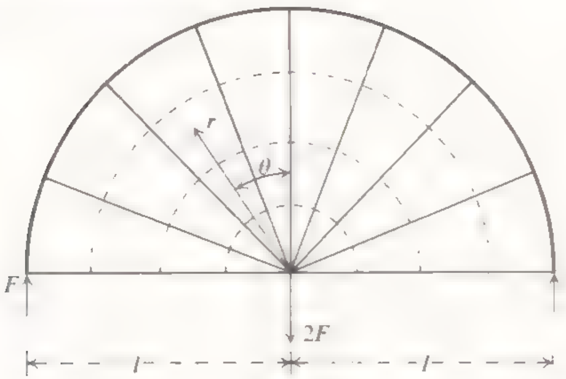


Fig. 4.5

$r = l$ and a complete fan of radii $-\pi/2 \leqslant \theta \leqslant \pi/2$. Equation (4.34), applied to $r = l$, gives $P = F$ and $(\mathbf{T}_1)_{r=l} = P = F$. The second of (4.33) and (4.35) gives $\mathbf{T}_2 = 0$ and hence, by (4.32), $t_2 = 0$. The first of (4.33) gives $\mathbf{T}_1$ independent of α and therefore equal to F. Equation (4.32) then gives $t_1 = F/\sigma_T r$. The fact that t_1 and t_2 are non-negative and that the member at $r = l$ is in compression shows that the conditions of (4.3) are satisfied. The total volume of material in the structure is

$$V_{\min} = \pi l F(1/\sigma_T + 1/\sigma_C), \tag{4.56}$$

which may also be found by applying (4.6). The displacement at each of the forces F is, by (4.55), equal to $\sigma\epsilon(1/\sigma_T + 1/\sigma_C)l\pi/2$ in the direction of these forces. The point of application of the force $2F$ has zero displacement in (4.55). It follows that $W/\sigma\epsilon$ is equal to $V_{\min}$ in (4.56).

A Michell strain field, which is valid for the whole plane, can be constructed by joining two opposed systems of circles and radii of half-angle $\theta_0 \leqslant \pi/4$, with strains of opposite sign, to two systems of orthogonal straight lines and two regions of hydrostatic compression (Fig. 4.6). The displacements in the upper fan are given by (4.55); those in the lower fan can be obtained by an interchange of strain limits. The displacements in the region of orthogonal straight lines, referred to axes $O(x, y)$ in Fig. 4.6, are given by

$$u_x = \sigma\epsilon x/\sigma_T + \sigma\epsilon(1/\sigma_T + 1/\sigma_C)\theta_0 y, \quad u_y = -\sigma\epsilon(1/\sigma_T + 1/\sigma_C)\theta_0 x - \sigma\epsilon y/\sigma_C. \tag{4.57}$$

These follow from (4.38) with $u_0 = v_0 = 0$ and ω_0 adjusted so as to give continuity of displacements at $\theta = \theta_0$ or $y = 0$. The displacements in the regions of hydrostatic compression, which are referred to polar coordinates (ρ, ψ) in Fig. 4.6 are given by

$$u_\rho = -\sigma\epsilon\rho/\sigma_C, \quad u_\psi = -\sigma\epsilon(1/\sigma_T + 1/\sigma_C)\theta_0\rho. \tag{4.58}$$

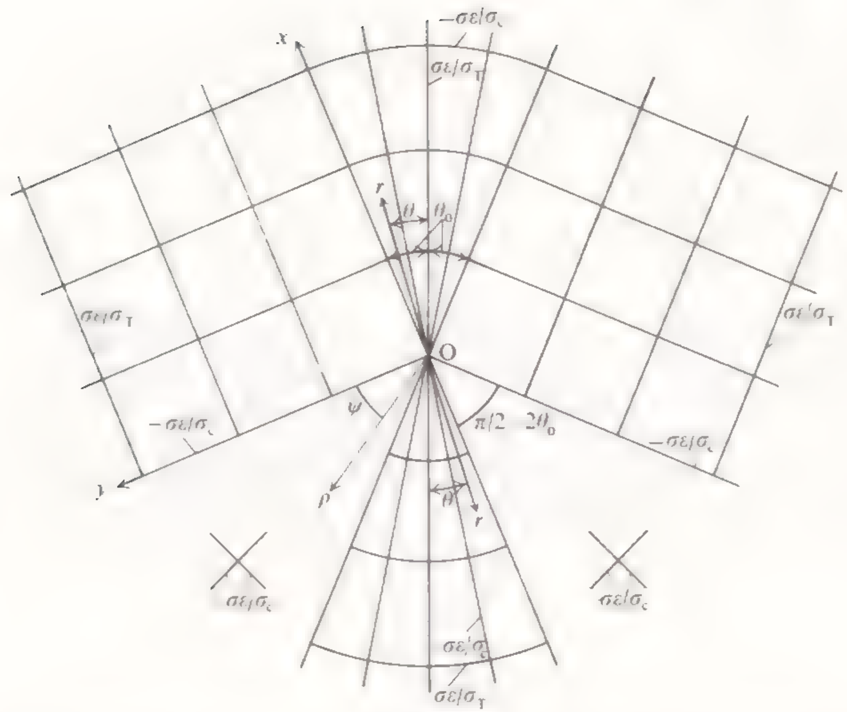


Fig. 4.6

The first equation of (4.58) gives a uniform dilatation of linear strain $-\sigma\epsilon/\sigma_C$ centred on O, while the second gives a rotation about O. Continuity with the displacements of (4.57) is satisfied at $\psi = 0$ or $x = 0$. The displacements of (4.58) match the displacement in the lower fan at $\theta = -\theta_0$, which are given by (4.55) with the signs reversed and σ_T interchanged with σ_C.

One interesting feature of the present strain field is the existence of lines whose points have no motion at right angles to the line of symmetry $\theta = 0$. These are given by either

$$\text{(a) } u_x \sin\theta_0 + u_y \cos\theta_0 = 0 \quad \text{or} \quad \text{(b) } u_\rho \cos(\psi + \theta_0) - u_\psi \sin(\psi + \theta_0) = 0. \tag{4.59}$$

Substitution from (4.57, 58) in (4.59) gives

$$\left.\begin{aligned} &\text{(a) } x/y = \{(\sigma_T + \sigma_C)\theta_0 \sin\theta_0 - \sigma_T \cos\theta_0\}/\{(\sigma_T + \sigma_C)\theta_0 \cos\theta_0 - \sigma_C \sin\theta_0\}, \\ &\text{(b) } \tan(\psi + \theta_0) = \sigma_T/(\sigma_T + \sigma_C)\theta_0, \end{aligned}\right\} \tag{4.60}$$

which define straight lines through O. If their direction is determined by χ, the angle made with the normal to $\theta = 0$, then since

$$\text{(a) } \chi = \theta_0 - \tan^{-1}(x/y), \qquad \text{(b) } \chi = \psi + \theta_0, \tag{4.61}$$

equations (4.60) give

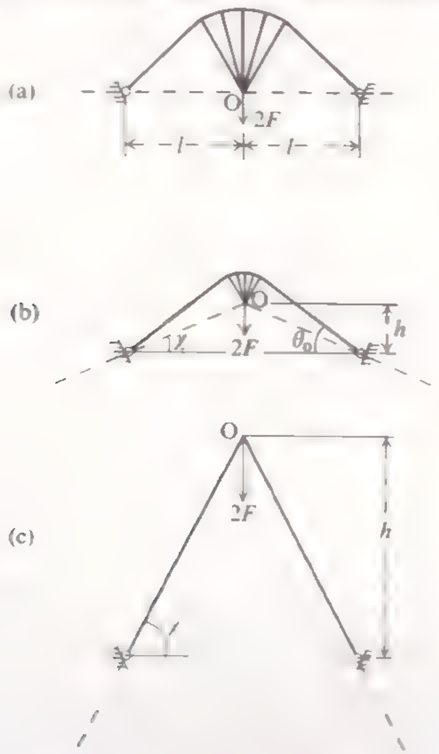
$$\left.\begin{array}{ll}\text{(a)} & \tan\chi = \{\sigma_T\cos^2\theta_0 - \sigma_C\sin^2\theta_0\}/(\sigma_T+\sigma_C)\{\theta_0 - (1/2)\sin 2\theta_0\},\\ \text{(b)} & \tan\chi = \sigma_T/(\sigma_T+\sigma_C)\theta_0.\end{array}\right\} \quad (4.62)$$

Relations (a) are valid for $\pi/4 \geqslant \theta_0 \geqslant \gamma$, where, as is easily seen from (4.60),

$$\tan\gamma = \sigma_T/(\sigma_T+\sigma_C)\gamma. \quad (4.63)$$

Relations (b) are valid for $\gamma \geqslant \theta_0 > 0$. For the special case when $\sigma_T = \sigma_C$ the value of γ is 37·4°.

The strain field of Fig. 4.6 may be used to construct solutions to the problems illustrated in Fig. 4.7. A force $2F$ acts along the perpendicular bisector of the segment of length $2l$ joining two fixed points. The point of action can be at any distance h from this segment. The form of the optimum structure depends upon the angle $\chi = \tan^{-1}(h/l)$, which must be identical with the χ in (4.62), since the line (shown dotted in Fig. 4.7), whose points have no displacement normal to the line of symmetry of the structures, must pass through the fixed supports.



(a) $\theta_0 = \pi/4$. (b) $\pi/4 > \theta_0 > \gamma$. (c) $\gamma > \theta_0$

Fig. 4.7

The value of θ_0 follows from (4.62) and the form of the structure depends upon whether θ_0 is greater than or equal to γ or less than γ. Examples of the first case are shown in Fig. 4.7 (a) and (b); the second case is shown in (c).

Inspection of the structures of Fig. 4.7 shows that the signs of the stresses in the various members agree with the signs of the virtual strain field of Fig. 4.6. The condition of zero displacement at the supports parallel to the line of symmetry of the structures is not satisfied by (4.55, 57, 58), but the addition of a suitable translation parallel to that line of symmetry will bring these supports to rest. The conditions of (4.3) for an optimum structure are or can be satisfied.

The volume of material in the structures is most easily obtained using (4.6), at least when $\theta_0 \geqslant \gamma$. The coordinates of a point of support are given by

$$x = l\sec\chi\sin(\theta_0-\chi), \quad y = l\sec\chi\cos(\theta_0-\chi). \quad (4.64)$$

Equations (4.57, 64) then give the displacements of this point, in particular, the displacement parallel to $\theta = 0$ in Fig. 4.6, which is $u_x\cos\theta_0 - u_y\sin\theta_0$. This is the displacement of the force $2F$, when the supports are taken to be at rest in the virtual deformation. It follows that

$$V_{\min} = 2F(u_x\cos\theta_0 - u_y\sin\theta_0)/\sigma\epsilon, \quad (4.65)$$

where u_x, u_y are to be taken from (4.57) and to be evaluated for values of x, y given by (4.64). The result, after some transformation, using the first of (4.62), is

$$V_{\min} = 2Fl\{(\sigma_T+\sigma_C)^2\theta_0^2 - \sigma_T\sigma_C\}/\sigma_T\sigma_C(\sigma_T+\sigma_C)\{\theta_0 - (1/2)\sin 2\theta_0\} \quad (4.66)$$
$$(\theta_0 \geqslant \gamma \text{ or } h/l \leqslant \tan\gamma),$$

which must be taken in conjunction with the first of (4.62), with $\tan\chi = h/l$. The result for the case $\theta_0 \leqslant \gamma$ is easily obtained by direct calculation, which gives

$$V_{\min} = (2Fl/\sigma_C)(l/h + h/l) \qquad (\theta_0 \leqslant \gamma \text{ or } h/l \geqslant \tan\gamma). \quad (4.67)$$

The figure which is obtained from the Hencky net of Fig. 4.6 by taking $\theta_0 = \pi/4$ is of great importance in its own right. It is reproduced in Fig. 4.8. It can be used to generate optimum structures for systems of parallel forces, such as are illustrated in Fig. 4.9. These structures are all determinate and it is clear by inspection that the signs of the end loads and stresses agree with those of the virtual strains of Fig. 4.8. The volume of material required for cases (a) and (b) can be calculated by observing that the x, y coordinates of the point of application of a force F are

$$x = (l-h)/\sqrt{2}, \quad y = (l+h)/\sqrt{2}, \quad (4.68)$$

and that the displacement of this point resolved in the direction of F is $(u_x - u_y)/\sqrt{2}$. The volume is thus

$$V_{\min} = \sqrt{2}F(u_x - u_y)/\sigma\epsilon. \quad (4.69)$$

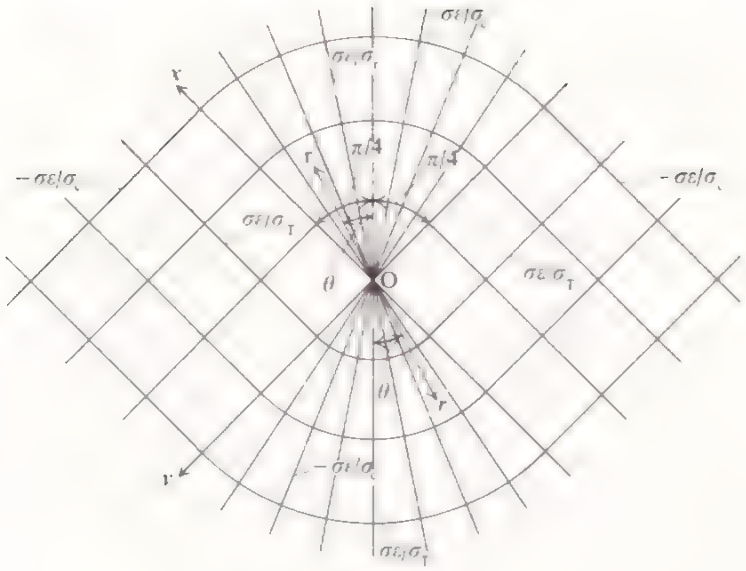


Fig. 4.8

where u_x and u_y are given by (4.57) with $\theta_0 = \pi/4$ and x and y by (4.68). The resulting formula for V_{min} is thus

$$V_{min} = (Fl/\sigma_T\sigma_C)\{(\sigma_T + \sigma_C)(1 + \pi/2) + (\sigma_T - \sigma_C)h/l\}. \qquad (4.70)$$

The special case of (a) where $h = 0$ should be compared with (4.56). The structure of Fig. 4.9 (a) has a volume $(1 + \pi/2)/\pi$ times that of Fig. 4.5 and is thus some 18 per cent lighter, although it carries the same forces. The reason lies in the fact that the strain field upon which Fig. 4.5 is based is, unlike that of Fig. 4.8, not valid in the whole plane. Fig. 4.5 does not give an absolute optimum, but only that for the plane cut along a ray through the origin, so that the displacements are single valued.

The volume of material for case (c) of Fig. 4.9 follows by observing that the coordinates of the point of application of a F are

$$r = \sqrt{(h^2 + l^2)}, \qquad \theta = \tan^{-1}(l/h), \qquad (4.71)$$

and that the displacement of this point is given by $-u_r \cos\theta + u_\theta \sin\theta$, where u_r and u_θ are given by (4.55), with their signs reversed and σ_T and σ_C interchanged. The resulting formula for V_{min} is thus

$$V_{min} = (2Fl/\sigma_T\sigma_C)\{(\sigma_T + \sigma_C)\tan^{-1}(l/h) + \sigma_T h/l\}. \qquad (4.72)$$

A three-dimensional strain field can be generated from Fig. 4.6 by a rotation about the line of symmetry $\theta = 0$. The original two-dimensional strain field is imposed upon all the co-axial planes so obtained, to produce an axially symmetric field. The only 'new' strain component induced by this process is a hoop strain ϵ_H given by

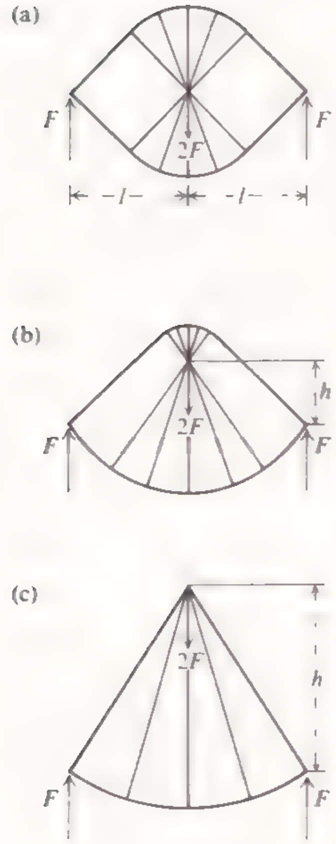


Fig. 4.9

$$\epsilon_H = (u_r \sin\theta + u_\theta \cos\theta)/r \sin\theta, \quad (u_x + u_y)/(x + y) \quad \text{and}$$
$$\{u_\rho \cos(\theta_0 + \psi) - u_\psi \sin(\theta_0 + \psi)\}/\rho \cos(\theta_0 + \psi), \qquad (4.73)$$

in the various regions of Fig. 4.6. Substitution from (4.55, 57, 58) gives

$$\epsilon_H = \sigma\epsilon/\sigma_T - \sigma\epsilon(1/\sigma_T + 1/\sigma_C)\theta \cot\theta \quad \text{and} \quad -\sigma\epsilon/\sigma_C + \sigma\epsilon(1/\sigma_T + 1/\sigma_C)\theta \cot\theta, \qquad (4.74)$$

$$\epsilon_H = \sigma\epsilon[x\{1/\sigma_T - (1/\sigma_T + 1/\sigma_C)\theta_0\} - y\{1/\sigma_C - (1/\sigma_T + 1/\sigma_C)\theta_0\}]/(x + y), \qquad (4.75)$$

and

$$\epsilon_H = -\sigma\epsilon/\sigma_C + \sigma\epsilon(1/\sigma_T + 1/\sigma_C)\theta_0 \tan(\theta_0 + \psi). \qquad (4.76)$$

It then follows, since $0 \leqslant \theta \leqslant \theta_0 \leqslant \pi/4$, $x, y \geqslant 0$ and $\theta_0 \leqslant \theta_0 + \psi \leqslant \pi/2 - \theta_0$, that

$$-\sigma\epsilon/\sigma_C \leqslant \epsilon_H \leqslant \sigma\epsilon/\sigma_T, \tag{4.77}$$

for all the expressions (4.74, 75, 76). This three-dimensional strain field has its strain limited as is required by (4.3) and so can be used to generate optimum structures. The structures of Figs. 4.7 and 4.9 can be rotated about their lines of symmetry and any number of the resulting structures and loading systems can be combined together to produce optimum structures in three dimensions. Two simple cases are illustrated in Fig. 4.10. The solution for (a) is obtained by

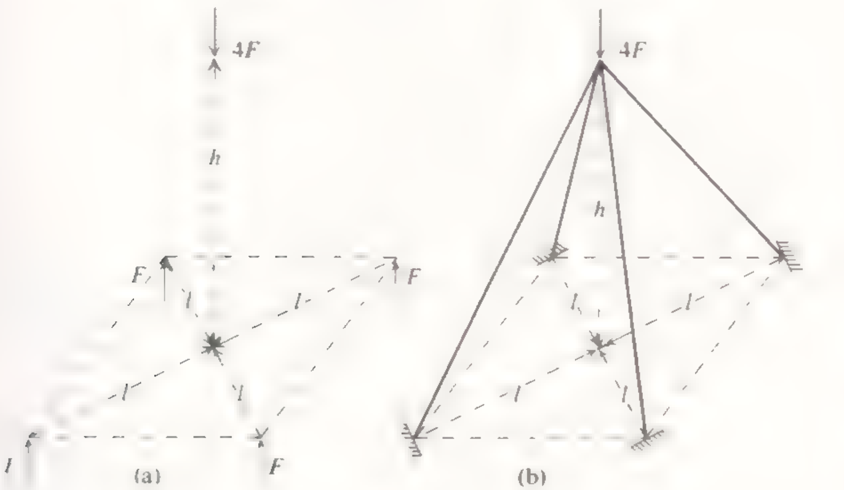


Fig. 4.10

placing one of the figures from Fig. 4.9 in each vertical plane through the diagonals of the square formed by the points of application of the forces F. The solution for (b) is obtained in the same way using one of the figures from Fig. 4.7. The result of using Fig. 4.7 (c) is shown.

4.6 Equiangular spirals

An interesting strain field may be obtained from (4.43) by taking $a = b = 1$ and making $A = B$. This gives

$$A = B = \sqrt{2}\,.R \exp(\alpha + \beta), \tag{4.78}$$

where R is a constant. It is convenient to take $\phi_0 = -\pi/4$ and then (4.42) gives

$$\phi = -\pi/4 - \alpha + \beta. \tag{4.79}$$

Equation (4.13) gives, with $x_0 = R$, $y_0 = 0$,

$$x + iy = r\exp(i\theta) = R\exp\{\alpha + \beta + i(\beta - \alpha)\}, \tag{4.80}$$

where polar coordinates r, θ have been introduced. Equation (4.80) yields

$$r = R\exp(\alpha + \beta), \qquad \theta = \beta - \alpha \pmod{2\pi}. \tag{4.81}$$

The coordinate lines have the following equations:

$$\left.\begin{array}{ll} \alpha\text{-lines} & r = R\exp(2\beta - \theta), \\ \beta\text{-lines} & r = R\exp(2\alpha + \theta). \end{array}\right\} \tag{4.82}$$

These are orthogonal equiangular spirals, with angles $\pi/4$, encircling the origin (Fig. 4.11).

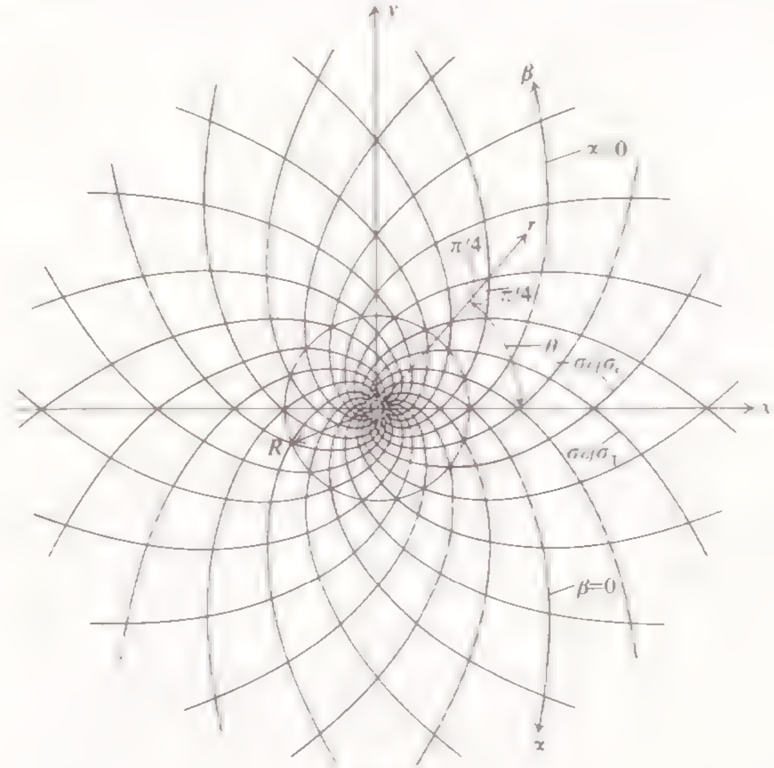


Fig. 4.11

The displacements take a particularly simple form when $\sigma_T = \sigma_C$. This assumption will therefore be made for the rest of the section. Equation (4.51) gives, with $\omega_0 = -\epsilon$

$$\omega = -\epsilon - 2\epsilon(\alpha + \beta). \tag{4.83}$$

Equation (4.19), with $u_0 = v_0 = 0$, gives

$$u + iv = (u_r + iu_\theta)\exp i(\theta - \phi) = \sqrt{2}\,.\,\epsilon R(1-i)(\alpha+\beta)\exp(\alpha+\beta), \quad (4.84)$$

where the polar components u_r, u_θ have been introduced. Equations (4.84) and (4.81) then yield

$$u_r = 0, \quad u_\theta = -2\epsilon r \log(r/R), \quad (4.85)$$

which shows that the virtual deformation consists of the rotation of concentric circles, with a circle of radius r rotating through an angle $-2\epsilon\log(r/R)$. It is to be remarked that the circle $r = R$ has zero rotation and can be taken as a fixed support.

The present strain field was applied by Michell to the problem of the cantilever (Fig. 4.12). An optimum structure is required to transmit the force F to the fixed circular boundary $r = R$. A solution is obtained by using the strain field of Fig. 4.11.

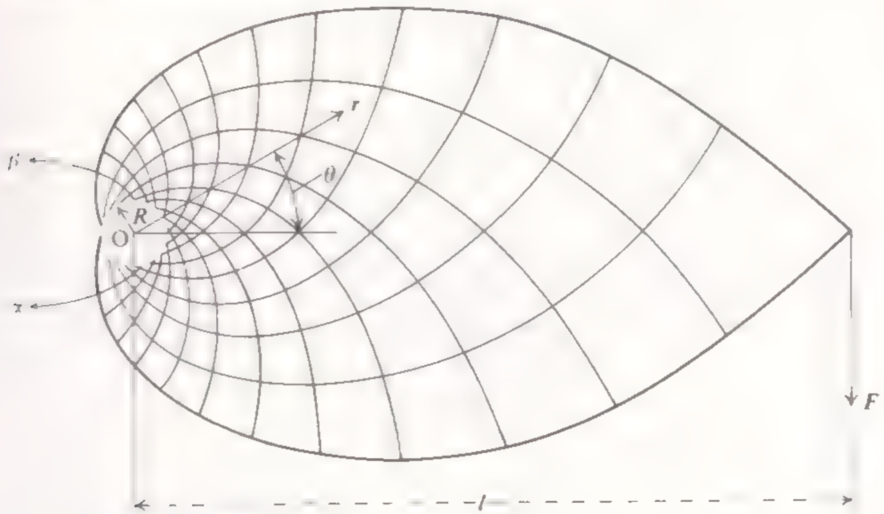


Fig. 4.12

The force F requires concentrated members to carry it. These members lie along the coordinate lines $\alpha, \beta = (1/2)\log(l/R)$, as is shown by (4.81). They carry forces of magnitude $F/\sqrt{2}$. Equations (4.34, 35) then give, recalling (4.79),

$$\mathbf{T}_2 = -F/\sqrt{2} \text{ at } \beta = (1/2)\log(l/R), \quad \mathbf{T}_1 = F/\sqrt{2} \text{ at } \alpha = (1/2)\log(l/R). \quad (4.86)$$

These are the boundary conditions for (4.52), which, with $a = b = 1$, can be written

$$\partial\mathbf{T}_2/\partial\beta = \mathbf{T}_1, \quad \partial\mathbf{T}_1/\partial\alpha = \mathbf{T}_2. \quad (4.87)$$

It is convenient to change to new coordinates $\bar{\alpha}$, $\bar{\beta}$ given by

$$\bar{\alpha} = (1/2)\log(l/R) - \alpha, \quad \bar{\beta} = (1/2)\log(l/R) - \beta. \quad (4.88)$$

Equations (4.86, 87) then become

$$\left.\begin{aligned} &\partial\mathbf{T}_2/\partial\bar{\beta} = -\mathbf{T}_1, \quad \partial\mathbf{T}_1/\partial\bar{\alpha} = \mathbf{T}_2, \\ \text{with} \quad &\mathbf{T}_2 = -F/\sqrt{2} \text{ at } \bar{\beta} = 0 \text{ and } \mathbf{T}_1 = F/\sqrt{2} \text{ at } \bar{\alpha} = 0. \end{aligned}\right\} \quad (4.89)$$

Equations (4.89) agree with (4.43) with $a = b = -1$ and $\mathbf{T}_2$ written for A and $\mathbf{T}_1$ for B. The required solution is then given by (4.50), which yields

$$\begin{aligned} \mathbf{T}_2(\bar{\xi}, \bar{\eta}) &= -(F/\sqrt{2})\, I_0\{2\sqrt{(\bar{\xi}\,\bar{\eta})}\} + \int_0^{\bar{\eta}} (-F/\sqrt{2})\, I_0[2\sqrt{\{(-\bar{\xi})(\bar{\beta} - \bar{\eta})\}}]\, d\bar{\beta} \\ &= -(F/\sqrt{2})\, I_0\{2\sqrt{(\bar{\xi}\bar{\eta})}\} - (F/\sqrt{2})\sqrt{(\bar{\eta}/\bar{\xi})}\, I_1\{2\sqrt{(\bar{\xi}\bar{\eta})}\}, \end{aligned} \quad (4.90)$$

where use has been made of the relation $I_1'(t) + (1/t)I_1(t) = I_0(t)$, between the modified Bessel functions of the zero and first orders.

The value of $\mathbf{T}_1$ follows by symmetry and is

$$\mathbf{T}_1(\bar{\xi}, \bar{\eta}) = -\mathbf{T}_2(\bar{\eta}, \bar{\xi}). \quad (4.91)$$

It follows by (4.32) that $t_1, t_2 \geqslant 0$ and from (4.36) that

$$\left.\begin{aligned} &t(\bar{\xi}, \bar{\eta}) = (F/\sigma R)\exp\{-(\xi+\eta)\}\,[I_0\{2\sqrt{(\bar{\xi}\bar{\eta})}\} + (\bar{\xi}+\bar{\eta})I_1\{2\sqrt{(\bar{\xi}\bar{\eta})}\}/2\sqrt{(\bar{\xi}\bar{\eta})}], \\ \text{where} \quad &\bar{\xi} = (1/2)\log(l/R) - \xi, \quad \bar{\eta} = (1/2)\log(l/R) - \eta. \end{aligned}\right\} \quad (4.92)$$

The total volume of material is best obtained using (4.6) and (4.85). The result is

$$V_{\min} = (2Fl/\sigma)\log(l/R). \quad (4.93)$$

An interesting application of the present solution to the problem of transmitting a pure bending moment is given in H. S. Y. Chan (1972). The structural layout and the given moments M, which are applied by rigid circles of radius R, are shown in Fig. 4.13. The virtual strain field is defined separately in each of the three regions: (a) $R \leqslant r \leqslant l/\sqrt{2}$, $0 \leqslant \theta \leqslant \pi$; (b) $x + y \leqslant l$, $y \geqslant 0$, less the region (a); (c) $x + y \geqslant l$, $x \leqslant l$. The deformation in the rest of the plane follows by symmetry. The displacement field is given by

$$\left.\begin{aligned} &\text{(a)}\ u_r = 0, \quad u_\theta = \epsilon r\,\{1 + 2\log(l/\sqrt{2}r)\}, \\ &\text{(b)}\ u_r = 0, \quad u_\theta = \epsilon r, \\ &\text{(c)}\ u_x = -\epsilon(l - x), \quad u_y = \epsilon(l - y). \end{aligned}\right\} \quad (4.94)$$

The field defined by (4.94) (a) differs from that defined by (4.85) by a rigid body rotation about the origin $r = 0$. It thus has the same strain field as Fig. 4.11, with $\sigma_T = \sigma_C = \sigma$. The field of (4.94) (b) is a rigid body rotation of amount ϵ

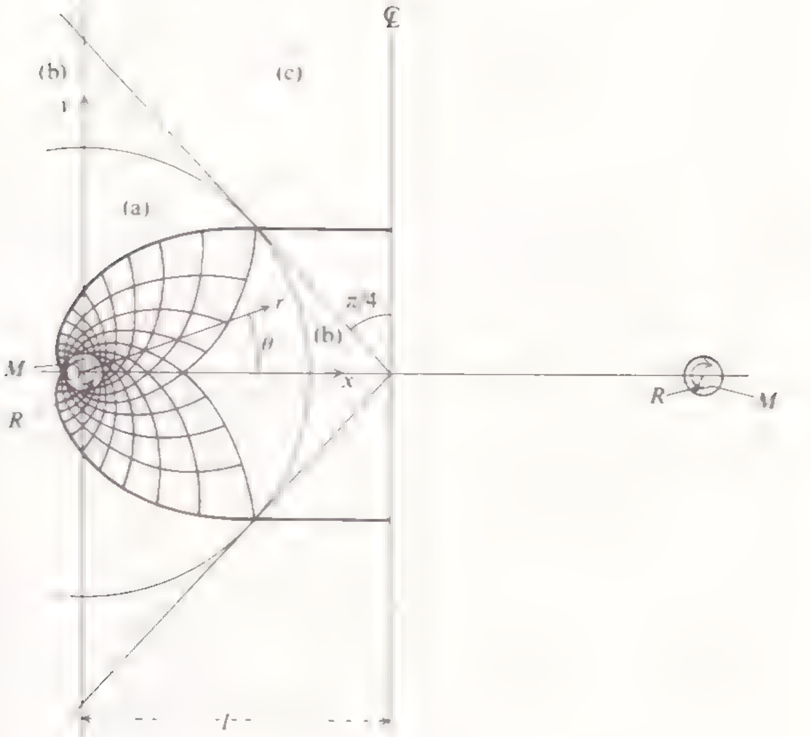


Fig. 4.13

about the origin and so has zero strain everywhere. The field of (4.94) (c) has lines of principal strain with strain ϵ parallel to Ox and lines of principal strain with strain $-\epsilon$ parallel to Oy.

The displacements are continuous at the boundary of regions (a) and (b), where $r = l/\sqrt{2}$. At the boundary between (b) and (c), the displacements of (b) have the values $u_x = -u_\theta \sin\theta = -\epsilon y$ and $u_y = u_\theta \cos\theta = \epsilon x$. On this boundary $x + y = l$ and so continuity with the displacements of (c) is satisfied. The symmetry conditions are clearly satisfied by (a) and (b) at $y = 0$, since (4.94) (a) and (b) are valid for $y < 0$ as well. The symmetry condition at $x = l$, namely $u_x = 0$, is also satisfied.

The strain field defined by (4.94) is thus a continuous Michell strain field and so the layout of Fig. 4.13 satisfies the kinematic conditions for an optimum structure. It remains only to show that it can carry the moments M with stresses agreeing in sign with the corresponding strains. The loads in the tie and strut at the centre line are M/l. These require concentrated members, which lie along $\pm\epsilon$ lines and enter the cantilevers of equiangular spirals in the directions of $\pm\epsilon$ strain lines respectively. The boundary conditions of (4.86) would require

modification, if detailed analysis was needed, but it is clear that a consistent solution for member sizes can be found.† The structure of Fig. 4.13 is thus an optimum structure, with a volume of material given by (4.6) and (4.94) as

$$V_{\min} = (2M/\sigma)\left\{1 + 2\ln\left(l/\sqrt{2}R\right)\right\}. \tag{4.95}$$

4.7 Michell's sphere

The strain field of Fig. 4.11 may be used to construct the optimum structure for the transmission of a uniformly distributed torque at an inner circular boundary to a similarly uniformly distributed torque at a concentric outer circular boundary (Fig. 4.14). Michell extended this solution to three-dimensions and showed that the optimum structure, which balances two opposing torques applied to two rigid co-axial circular discs, consists of a framework of members lying along the rhumb lines‡ on a spherical surface, which passes through the edges of the discs. Fig. 4.14 may be regarded as giving an end-view of this framework, showing one disc with its applied torque and the balancing forces from the second hemisphere.

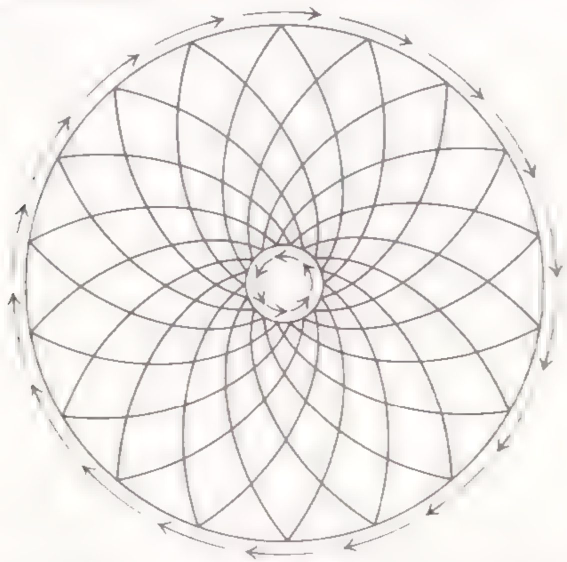


Fig. 4.14

It is convenient to use spherical polar coordinates (r, θ, ϕ) with $\theta = 0, \pi$ on the axis of the discs. Let the spherical surface have radius R and let the edges of the discs be defined by $r = R$, $\theta = \theta_0$ and $\pi - \theta_0$.

† The integral in (4.90) is absent here and $F/\sqrt{2}$ is replaced by M/L
‡ Lines which bisect the angles between the meridians and parallels in any spherical polar coordinate system.

A spherical shell, with middle surface at $r = R$, can be in equilibrium, when carrying only a shear stress resultant S, associated with the θ and ϕ directions, and depending upon θ alone. The conditions of equilibrium† give

$$\mathrm{d}S/\mathrm{d}\theta + 2S \cot\theta = 0, \tag{4.96}$$

which integrates to give

$$S = S_0 \sin^2\theta_0/\sin^2\theta, \tag{4.97}$$

where S_0 is the input of shear per unit length at the edges of the discs. The shear stress resultant S is equivalent to two principal stress resultants $\pm S$ lying along the rhumb lines and requiring end load carrying material of equivalent thicknesses $t_1 = t_2 = S/\sigma$, where the simplifying assumption that $\sigma_T = \sigma_C = \sigma$ has been made. The required total thickness t, is thus given by

$$t = t_1 + t_2 = 2S/\sigma = 2S_0 \sin^2\theta_0/\sigma \sin^2\theta.\ddagger \tag{4.98}$$

Consider now a virtual deformation defined by

$$u_r = 0, \quad u_\theta = 0, \quad u_\phi = 2\epsilon r \sin\theta \log\tan(\theta/2). \tag{4.99}$$

The corresponding components of strain can be calculated using the standard formulae.§ The only component which does not vanish is the shear strain $\gamma_{\theta\phi}$, which has the value 2ϵ. The corresponding principal strains are $\pm\epsilon$ and so the strain system defined by (4.99) is limited in the way required by Michell's conditions. These principal strains lie along the rhumb lines of all spheres centred at $r = 0$, including of course $r = R$. The signs of the principal strains agree with those of the principal stresses corresponding to S, since the sign convention for stresses and strains is chosen so this is true. Michell's sphere is thus the optimum structure for the transmission of torque.

The volume of material follows in the usual way from the virtual work. The axial rotation corresponding to u_ϕ is $u_\phi/r \sin\theta$. If the torque is T, the work is $2Tu_\phi/r \sin\theta$ evaluated at $\theta = \pi - \theta_0$. Hence

$$V_{\min} = (4T/\sigma) \log\cot(\theta_0/2). \tag{4.100}$$

The relation between T and S_0 is

$$T = 2\pi R^2 S_0 \sin^2\theta_0. \tag{4.101}$$

4.8 Cycloids

A particular solution to (4.43) with $a = -1$, $b = 1$ is given by

$$A = 2h\cos(\alpha+\beta), \qquad B = 2h\sin(\alpha+\beta), \tag{4.102}$$

where h is a constant. The corresponding formula for ϕ is given by (4.42). Taking $\phi_0 = 0$ yields

$$\phi = \alpha + \beta. \tag{4.103}$$

† Love (1927), Section 344
‡ This is the same as would be required for a shell with allowable stress in shear $= \sigma/2$.
§ Love (1927), Chap. 1, Section 22

Equation (4.13) now gives, with $x_0 = y_0 = 0$, on substituting from (4.102, 103), the formulae:

$$x = (h/2)\{\sin 2(\alpha+\beta) + 2(\alpha-\beta)\}, \qquad y = (h/2)\{1 - \cos 2(\alpha+\beta)\}, \tag{4.104}$$

which shows that the coordinate curves are cycloids (Fig. 4.15).† The effective coverage of this coordinate system is the strip $0 \leqslant y \leqslant h$. A vanishes for $\alpha + \beta = \pi/2$ and B for $\alpha + \beta = 0$ and the coordinate curves envelop these loci, which by (4.104) are seen to be the lines $y = h$ and $y = 0$ respectively.

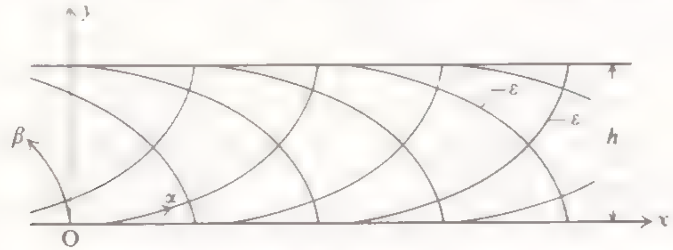


Fig. 4.15

The virtual displacements may now be calculated. Assuming for simplicity that $\sigma_T = \sigma_C = \sigma$, (4.51), with $\omega_0 = \pi\epsilon/2$, gives

$$\omega = \pi\epsilon/2 + 2\epsilon(\alpha - \beta), \tag{4.105}$$

and so (4.19), with $u_0 = v_0 = 0$, integrates to give

$$u_x + iu_y = (u + iv)e^{i\phi} = \epsilon h\{(\alpha - \beta + \pi/4)e^{2i(\alpha+\beta)} + \alpha + \beta - \pi/4 + i(\alpha-\beta)(\alpha - \beta + \pi/2)\}. \tag{4.106}$$

Equation (4.106) gives the Cartesian components (u_x, u_y) and (4.104) may then be used to express these in terms of (x, y). The result is

$$\left.\begin{aligned} u_x &= -\pi\epsilon y/2 + \epsilon h\Big((1/2)\cos^{-1}(1 - 2y/h) \\ &\qquad + [x/h - \sqrt{\{(y/h)(1 - y/h)\}}](1 - 2y/h)\Big), \\ u_y &= \pi\epsilon x/2 + \epsilon h\{(x/h)^2 - (y/h)(1 - y/h)\}, \end{aligned}\right\} \tag{4.107}$$

where the inverse cosine is to be taken between 0 and π. It is to be remarked that both u_x and u_y vanish at $x = 0, y = 0$ and at $x = 0, y = h$.

Equations (4.52) have a solution, when $a = -1$, $b = 1$, of the form

$$T_1 = 2F\cos(\alpha+\beta), \qquad T_2 = -2F\sin(\alpha+\beta). \tag{4.108}$$

† This is the Hencky net used by Prandtl to analyse the plastic deformation of a strip under transverse pressure. See Hill (1950), VIII 5.

Equations (4.32), with $\sigma_T = \sigma_C = \sigma$, then give

$$t_1 = (F/\sigma h)\cot(\alpha+\beta), \quad t_2 = (F/\sigma h)\tan(\alpha+\beta), \tag{4.109}$$

which are positive $0 < \alpha + \beta < \pi/2$ and thus show that the signs of (4.108) agree with those of the virtual strains of (4.107). They become unbounded at, respectively, $y = 0$ and h. This is an indication that concentrated members are required along these lines and, for consistency with the strains, a tension member must lie along $y = 0$ and a compression member along $y = h$.

The Cartesian stress resultants T_x and S_{xy} can be calculated from (4.108) in the usual way by considering an infinitesimal right-angled triangle, with hypotenuse of length dy parallel to Oy. This gives

$$T_x\,dy = \mathbf{T}_1\,d\beta\cos\phi + \mathbf{T}_2\,d\alpha\sin\phi, \quad S_{xy}\,dy = \mathbf{T}_1\,d\beta\sin\phi - \mathbf{T}_2\,d\alpha\cos\phi, \tag{4.110}$$

where dα and dβ are related so that d$x = 0$. Substitution from (4.103, 108) gives, making use of (4.104),

$$T_x = 2F\,d\sqrt{\{(y/h)(1-y/h)\}}/dy, \quad S_{xy} = F/h. \tag{4.111}$$

The constant shear flow S_{xy} requires balancing at $y = 0, h$. This will be assumed to be done by concentrated members lying along these lines.

The resultant force components and couple corresponding to (4.111) are given by

$$\int_0^h T_x\,dy = 0, \quad \int_0^h S_{xy}\,dy = F \quad \text{and} \quad \int_0^h T_x(-y)\,dy = \pi Fh/4. \tag{4.112}$$

This indicates the loads that must be carried by the optimum structure, which carries the stresses (4.108) and 'allows' the strain field (4.107).

The problem to which the present investigation will be applied is the optimum design of the 'shear bracing' of a long cantilever of length $l \gg h$ and uniform depth h under a tip shear F (Fig. 4.16). The unwanted bending

Fig. 4.16

moment of (4.112) is cancelled out by forces $\pi F/4$ applied to the flanges at the tip. External forces are provided at the root to balance the loads in the web. The 'errors' in the applied loading will only affect the volume of the structure by an amount of order Fh/σ, which is small compared to the web volume of order Fl/σ.

Michell's conditions of (4.3) are satisfied everywhere except in the flanges up to a distance $\pi h/4$ from the tip. This discrepancy will produce an error of order $F\epsilon h$ in the virtual work or Fh/σ in the volume, which can be neglected. The required volume is given by

$$\sigma\epsilon V_{\min} = \int_0^h \left\{T_x [u_x]_{x=0}^{x=l} + S_{xy}[u_y]_{x=0}^{x=l}\right\} dy + (\pi F/4)\,\left[(u_x)_{x=l}\right]_{y=0}^{y=h}. \tag{4.113}$$

Substituting from (4.107, 111) in (4.113) then yields

$$V_{\min} = Fl^2/\sigma h + \pi Fl/2\sigma. \tag{4.114}$$

The first term of (4.114) is the amount of material required for the flanges of the girder irrespective of the nature of the shear bracing. The second term gives the material required for the shear bracing itself. In the present solution that shear bracing consists of a continuous array of orthogonal cycloids (Fig. 4.15).

The optimum shear bracing, which consists of straight members joining flange to flange, may be shown to be given by the Warren truss, in which members are inclined at an angle of $\pi/4$ to the flanges. The volume for the Warren truss is

$$V = Fl^2/\sigma h + 2Fl/\sigma \quad \text{(Warren truss)}, \tag{4.115}$$

and this has $4/\pi$ times the material in the shear bracing as that of (4.114).

A closer approximation to the layout of cycloids is shown in Fig. 4.17. This has a volume given by

$$V = Fl^2/\sigma h + \sqrt{3}Fl/\sigma \quad \text{(Truss of Fig. 4.17)}, \tag{4.116}$$

with a volume for the shear bracing intermediate between that of (4.114) and (4.115).

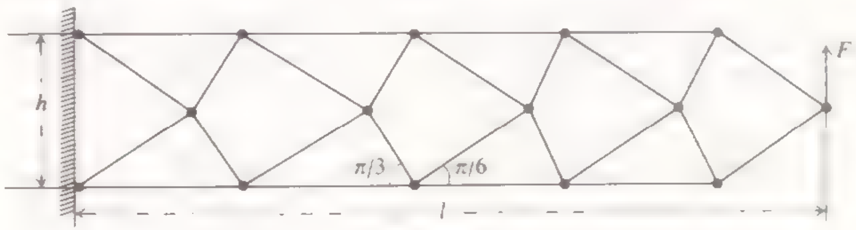


Fig. 4.17

4.9 Fields developed from circles

It is possible to develop a Hencky net beginning with two orthogonal circles as starting curves β and $\alpha = 0$ (Fig. 4.18).† Taking $a = b = 1$ and $\phi_0 = 0$ in (4.42) gives

$$\phi = -\alpha + \beta. \tag{4.117}$$

† This Hencky net has been much used in plastic flow investigations. See for example Hill (1950) Chap. VII.

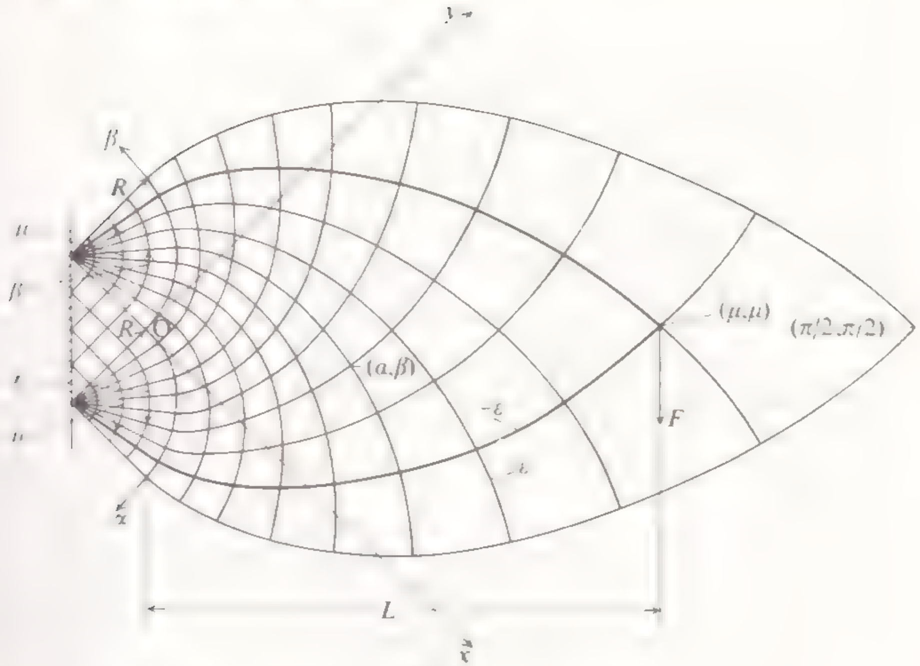


Fig. 4.18

The boundary conditions for A and B are

$$A = R \text{ on } \beta = 0, \qquad B = R \text{ on } \alpha = 0, \tag{4.118}$$

and with these (4.50) integrates, just as (4.90) did, to give

$$A(\alpha, \beta) = B(\beta, \alpha) = R\,[I_0\{2\sqrt{(\alpha\beta)}\} + \sqrt{(\beta/\alpha)}\, I_1\{2\sqrt{(\alpha\beta)}\}]. \tag{4.119}$$

Equation (4.13) may be integrated in terms of Lommel's functions (Chan, 1967), but the results are somewhat involved. Attention will be limited here to points for which $x = y = L/\sqrt{2}$ and $\alpha = \beta = \mu$. For this (4.13), with $x_0 = y_0 = 0$, gives

$$L = (R/\sqrt{2}) \int_0^{2\mu} \{I_0(t) + I_1(t)\}\, dt = (R/\sqrt{2})\, \{I_0(2\mu) - 1 + 2 \sum_{n=0}^{\infty} (-1)^n I_{2n+1}(2\mu)\}, \tag{4.120}$$ †

from which values of L/R for various μ can be obtained.‡

The virtual strain field can now be determined. The assumption will again be made that $\sigma_T = \sigma_C = \sigma$ and so the principal strains are $\pm\epsilon$. The Hencky net generated by the circles β and $\alpha = 0$ is extended backwards, for negative values of α and β, by means of orthogonal circles and radii, to the centres of these

† The standard results $I_1(t) = I_0'(t)$, $I_{n-1}(t) + I_{n+1}(t) = 2I_n'(t)\ (n = 1, 2 \ldots)$ have been used here

‡ Hencky nets may be constructed graphically. See Hill (1950) Chap. VI, 5.

circles. The remaining triangular region in Fig. 4.18 is filled in with orthogonal straight lines. The boundary segment along $x + y + R = 0$ is taken as a fixed support. This is consistent with (4.25), which requires, in this case, the coordinate curves to meet it at an angle of $\pi/4$.

Equations (4.19, 21) can now be applied. These are valid for the whole region occupied by the Hencky net, including the fans and the triangle, because (4.18, 20) can be integrated along any path to give (4.19, 21) in spite of discontinuities in A and B, since ϕ, (u, v) and ω are continuous everywhere. Equation (4.25) gives $\omega = -\epsilon$ on the fixed segment, since $dx = -dy$ on the boundary, and (4.21) gives $\omega = \omega_0 = -\epsilon$ in the triangle. Equation (4.51) then gives

$$\omega = -\epsilon - 2\epsilon(\alpha + \beta) \tag{4.121}$$

in the main field $\alpha, \beta \geqslant 0$. The values of $u_0 = \epsilon R$, $v_0 - \epsilon R$ follow from the extension of the sides of the triangle. Equation (4.19) then gives, using (4.121),

$$u + iv = \epsilon R\,[\{1 - i + 2(\alpha - i\beta)\}\, I_0\{2\sqrt{(\alpha\beta)}\} + 2(1 - i)\sqrt{(\alpha\beta)}\, I_1\{2\sqrt{(\alpha\beta)}\}]. \tag{4.122}$$

The simplest proof of (4.122) is by direct verification.†

The strain field of Fig. 4.18 may be applied to solve the problem of the optimum cantilever supported on a fixed segment for any system of loads which give tension members along α-lines and compression members along β-lines. For the special case of a force F applied at $\alpha = \beta = \mu$ parallel to the rigid line of support, the required volume is, by (4.122), given by

$$V_{\min} = (\sqrt{2}FR/\sigma)\, \{(1 + 2\mu)\, I_0(2\mu) + 2\mu\, I_1(2\mu)\}. \tag{4.123}$$

The concentrated members required to carry F are shown in Fig. 4.18 and the required structure is enclosed within them. The circles of the fans and the orthogonal straight lines are not required in this case.

The strain field of Fig. 4.18 may be extended beyond $\alpha = \pi/2$ and $\beta = \pi/2$ by just increasing the starting circles. However the layout so obtained will not be valid for the whole plane. It will break down for the same reason as does the complete array of concentric circles and radii, namely, discontinuity of displacement. An extension of Fig. 4.18 to the whole plane is possible, nevertheless, and a quadrant of this is shown in Fig. 4.19. The arc AB is a starting circle with centre C. The region ABC is a right-angled fan of circles and radii and the region ACD one of orthogonal straight lines. The arc AB and its 'image' generates the region ABE, just as in Fig. 4.18. The straight segment AD and the curve AE generate a Hencky net in ADFE consisting of straight lines and involutes. This can be obtained using (4.26, 27), since B and hence $F(\beta)$ are known on the starting curve AE. The curve DF and its 'image' can now be used

† Equation (4.122) was first given in A. S. L. Chan (1960) where he applied it to the problem of the cantilever treated in this section.

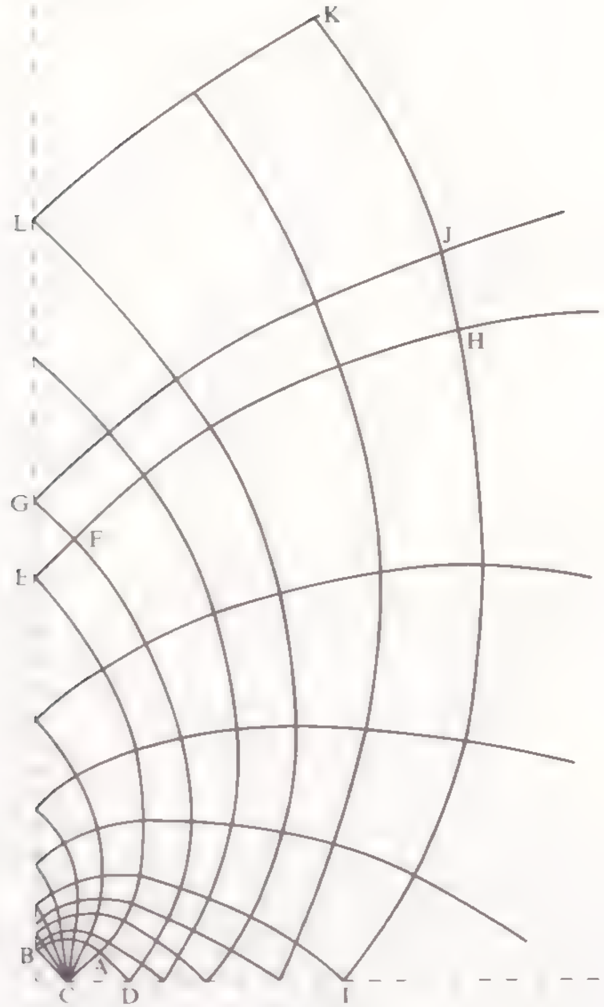


Fig. 4.19

to develop a Hencky net in a region, part of which, namely DFHI, is shown in Fig. 4.19.† The region EFG must be filled in with orthogonal straight lines and a region FGJH of straight lines and involutes can then be generated. The curve GJ and its 'image' now generates a new part of the field. The way in which this process of extension of the strain field over increasing regions of the plane can be carried out is now clear, as is the fact that it has no limit.

This extended field of Fig. 4.19 has been used by H. S. Y. Chan in his D.Phil. thesis to generate optimum structures for forces parallel or normal to a fixed

† This can be generated using (4.50), but the boundary conditions are now more complicated than (4.118).

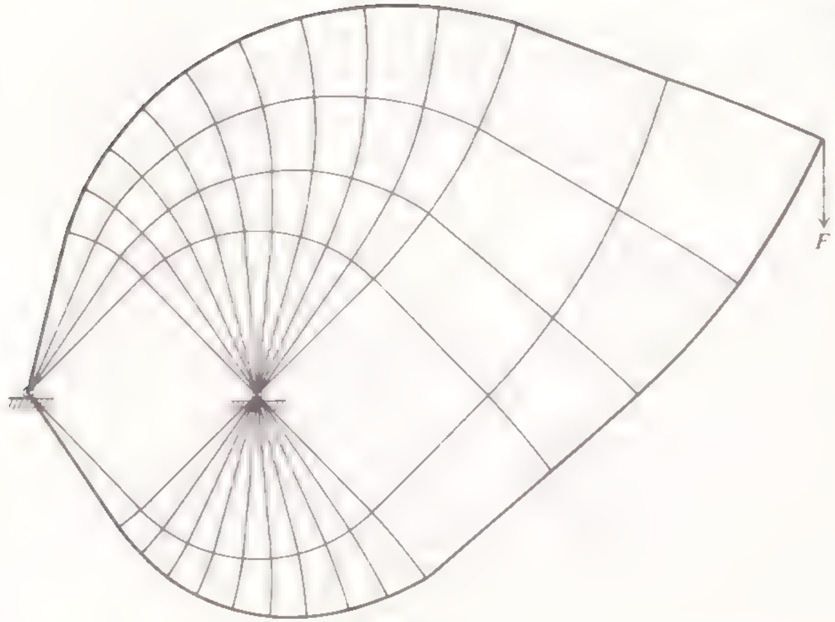


Fig. 4.20

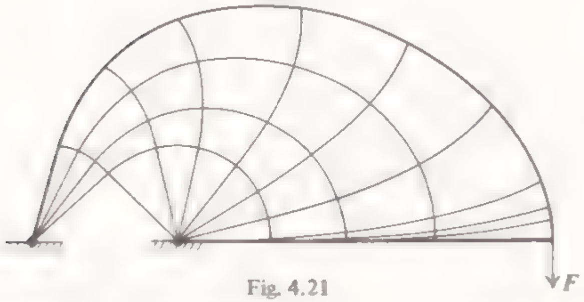


Fig. 4.21

line of support and applied at any point of the plane. He has also, in H. S. Y. Chan (1967), solved the same problem for structures restricted to lie in a half-plane. Examples of these important developments are shown in Figs. 4.20 and 4.21. It is to be remarked that, in Fig. 4.21, one set of layout lines are tangential to the boundary of the half-plane; this may be compared with the layout of cycloids in Fig. 4.15, where the structure is confined to a finite strip.

5 Plates loaded in their planes

5.1 Formulation of the problem

The present problem is to be interpreted as the design of plates of variable thickness t occupying a given region R of a plane referred to axes $O(x, y)$. The plates are loaded in R by forces with components (X, Y) per unit area and on a part B_F of the boundary of R by forces with components (X_n, Y_n) per unit length. The rest of the boundary B_C is rigidly constrained.

The state of stress in the plate is specified by stress resultants T_x, T_y and S, with reference to the axes $O(x, y)$ and by principal stress resultants T_1 and T_2, where T_1 is inclined at an angle ϕ to Ox. The stress distribution is in general continuous, but discontinuities at a finite number of lines are possible. The two specifications of stress are related by

$$\left.\begin{aligned} T_x &= (T_1 + T_2)/2 + \{(T_1 - T_2)/2\} \cos 2\phi, \\ T_y &= (T_1 + T_2)/2 - \{(T_1 - T_2)/2\} \cos 2\phi, \\ S &= \{(T_1 - T_2)/2\} \sin 2\phi. \end{aligned}\right\} \tag{5.1}$$

Equilibrium requires that

$$\partial T_x/\partial x + \partial S/\partial y + X = 0, \qquad \partial S/\partial x + \partial T_y/\partial y + Y = 0 \tag{5.2}$$

in R and, if (l, m) is the unit normal to the boundary,

$$lT_x + mS = X_n, \qquad lS + mT_y = Y_n \tag{5.3}$$

on the boundary, where (X_n, Y_n) are given forces on B_F and can be interpreted as forces of constraint on B_C. A final condition of equilibrium is that

$$lT_x + mS, \quad lS + mT_y \quad \text{are continuous at lines of discontinuity,} \tag{5.4}$$

where (l, m) is now a unit normal to the line of discontinuity.

Let it be assumed that the safety of the plate is governed by Tresca's criterion, with a tensile yield stress σ. This can be expressed, in a similar manner to (3.4), as

$$\left.\begin{aligned} T_1 + p_1 &= \sigma t, & -T_1 + q_1 &= \sigma t, \\ T_2 + p_2 &= \sigma t, & -T_2 + q_2 &= \sigma t, \\ T_1 - T_2 + p_3 &= \sigma t, & -T_1 + T_2 + q_3 &= \sigma t, \end{aligned}\right\} \tag{5.5}$$

where

$$p_1,\ q_1,\ p_2,\ q_2,\ p_3,\ q_3 \geqslant 0. \tag{5.6}$$

Equations (5.5, 6) can also be represented by means of a hexagon in the manner of Fig. 3.2. The zeros of the slack functions p_i, q_i $(i = 1, 2, 3)$, which correspond to the sides of the hexagon, define a set of points at which yielding is just beginning. It will be assumed, for the optimum structures studied here, that this set consists, for each slack variable, of a finite number of regions of R. This is the natural generalization to two-dimensions of the characteristic property of one-dimensional optimum structures found in Chapters 2 and 3.

The optimum designs are chosen so as to make the volume V of material in the plate a minimum. This is expressed by

$$\min V = \iint_R t \, dx \, dy. \tag{5.7}$$

The mathematical problem of optimization is thus to choose the functions T_x, T_y, S, T_1, T_2, ϕ, t, p_i, q_i $(i = 1, 2, 3)$ so as to make V of (5.7) a minimum, subject to the constraints (5.1, 2, 3, 4, 5, 6). This is a problem of the calculus of variations, but here the unknown functions are functions of the two variables x, y.

5.2 Formal derivation of necessary conditions

A strict derivation of the necessary conditions for the minimum of (5.7), using the methods of Sections 2.1, 2.5, 2.6 and 3.1, runs into difficulties when trying to show that the equations for the variations and the equations for the Lagrangian multipliers can actually be solved. This is because the solution of boundary value problems for partial differential equations is a much more difficult matter than that for ordinary differential equations. The present section is devoted to purely formal processes, which are carried out as an act of faith. Fortunately they lead to a dual problem and to sufficient conditions for the minimum.

The Lagrangian for the problem of Section 5.1 can be written

$$\begin{aligned} V^* = (1/\sigma\epsilon)\iint_R \{&\sigma\epsilon t + \lambda_1(T_1 + p_1 - \sigma t) + \mu_1(-T_1 + q_1 - \sigma t) + \lambda_2(T_2 + p_2 - \sigma t) + \\ &+ \mu_2(-T_2 + q_2 - \sigma t) + \lambda_3(T_1 - T_2 + p_3 - \sigma t) + \mu_3(-T_1 + T_2 + q_3 - \sigma t) + \\ &+ u(\partial T_x/\partial x + \partial S/\partial y + X) + v(\partial S/\partial x + \partial T_y/\partial y + Y)\} \, dx\, dy - \\ &- (1/\sigma\epsilon)\int_{B_F} \{u_1(lT_x + mS - X_n) + v_1(lS + mT_y - Y_n)\} \, ds, \end{aligned} \tag{5.8}$$

where T_x, T_y, and S are given by (5.1), ϵ is the usual positive infinitesimal, $\lambda_i(x, y)$, $\mu_i(x, y)$ $(i = 1, 2, 3)$, $u(x, y)$, $v(x, y)$, $u_1(s)$ and $v_1(s)$ are Lagrangian

multipliers and s is the arc length of the boundary. Application of Green's theorem, recalling (5.4), gives

$$\iint_R \{u(\partial T_x/\partial x + \partial S/\partial y) + v(\partial S/\partial x + \partial T_y/\partial y)\}\,dx\,dy =$$

$$= \int_{B_F + B_C} \{u(lT_x + mS) + v(lS + mT_y)\}\,ds +$$

$$+ \int_D \{\Delta u(lT_x + mS) + \Delta v(lS + mT_y)\}\,ds -$$

$$- \iint_R \{T_x \partial u/\partial x + T_y \partial v/\partial y + S(\partial v/\partial x + \partial u/\partial y)\}\,dx\,dy, \qquad (5.9)$$

where $\int_D$ is taken over the lines of discontinuity and Δu, Δv are the increments of u, v on these lines. Substitution from (5.9) into (5.8) and the introduction of (5.1) gives

$$\begin{aligned} V^* = (1/\sigma\epsilon) \iint_R \Big(&\sigma\epsilon t + \lambda_1(T_1 + p_1 - \sigma t) + \mu_1(-T_1 + q_1 - \sigma t) + \lambda_2(T_2 + p_2 - \sigma t) + \\ &+ \mu_2(-T_2 + q_2 - \sigma t) + \lambda_3(T_1 - T_2 + p_3 - \sigma t) + \mu_3(-T_1 + T_2 + q_3 - \sigma t) - \\ &- [(T_1 + T_2)/2 + \{(T_1 - T_2)/2\} \cos 2\phi]\,(\partial u/\partial x) - \\ &- [(T_1 + T_2)/2 - \{(T_1 - T_2)/2\} \cos 2\phi]\,(\partial v/\partial y) - \\ &[\{(T_1 - T_2)/2\} \sin 2\phi] - (\partial v/\partial x + \partial u/\partial y) + uX + vY\Big)\,dx\,dy + \\ &+ (1/\sigma\epsilon) \int_{B_F} \{u - u_1)(lT_x + mS) + (v - v_1)(lS + mT_y) + u_1 X_n + v_1 Y_n\}\,ds + \\ &+ (1/\sigma\epsilon) \int_{B_C} \{u(lT_x + mS) + v(lS + mT_y)\}\,ds + \\ &+ (1/\sigma\epsilon) \int_D \{\Delta u(lT_x + mS) + \Delta v(lS + mT_y)\}\,ds. \end{aligned} \qquad (5.10)$$

The conditions that V^* of (5.10) should be a minimum can now be derived in the usual way. Remembering (5.6) and that $t \geqslant 0$, then gives, in R, that

$$\left.\begin{aligned} &\sum_{i=1}^{3} (\lambda_i + \mu_i) = \epsilon \quad (t > 0), \qquad \sum_{i=1}^{3} (\lambda_i + \mu_i) \leqslant \epsilon \quad (t = 0), \\ &\lambda_1 - \mu_1 + \lambda_3 - \mu_3 = \epsilon_1, \quad \lambda_2 - \mu_2 - \lambda_3 + \mu_3 = \epsilon_2, \quad (T_1 - T_2)\gamma = 0, \\ &\lambda_i = 0\ (p_i > 0), \quad \mu_i = 0\ (q_i > 0), \quad \lambda_i \geqslant 0\ (p_i = 0), \quad \mu_i \geqslant 0\ (q_i = 0) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, 2, 3), \end{aligned}\right\} \qquad (5.11)$$

where

$$\left.\begin{aligned} \epsilon_1 &= (1/2)(\partial u/\partial x + \partial v/\partial y) + (1/2)(\partial u/\partial x - \partial v/\partial y) \cos 2\phi + \\ &\quad + (1/2)(\partial v/\partial x + \partial u/\partial y) \sin 2\phi, \\ \epsilon_2 &= (1/2)(\partial u/\partial x + \partial v/\partial y) - (1/2)(\partial u/\partial x - \partial v/\partial y) \cos 2\phi - \\ &\quad - (1/2)(\partial v/\partial x + \partial u/\partial y) \sin 2\phi, \\ \gamma &= -(1/2)(\partial u/\partial x - \partial v/\partial y) \sin 2\phi + (1/2)(\partial v/\partial x + \partial u/\partial y) \cos 2\phi. \end{aligned}\right\} \qquad (5.12)$$

Further conditions are, on the boundary,

$$u_1 = u, \quad v_1 = v \text{ on } B_F, \quad u = v = 0 \text{ on } B_C, \qquad (5.13)$$

and on the lines of discontinuity, u, v are continuous functions

(5.14)

The conditions of (5.11, 12, 13, 14) may be given a kinematic interpretation. The functions u, v can be regarded as Cartesian virtual displacement components. Equations (5.12) then show that ϵ_1, ϵ_2 and γ are virtual strain components associated with the directions ϕ and $\phi + \pi/2$ of the principal stress resultants T_1 and T_2. Equations (5.13) show that (u_1, v_1) are the boundary values of (u, v) and that the displacements vanish on the constrained part of the boundary B_C. Equation (5.14) implies that the displacements are continuous everywhere. Finally (5.11) imposes restrictions on the virtual strains. If $T_1 \neq T_2$ then $\gamma = 0$ and ϵ_1, ϵ_2 are principal strains with principal directions coinciding with those of the principal stress resultants. If $T_1 = T_2$, then, by (5.1), ϕ is not required for the specification of the stress resultants and so may be chosen to make $\gamma = 0$ under these conditions as well. The remaining equations of (5.11) limit the magnitude of these principal strains.

5.3 The dual problem

The problem of Section 5.1 may now be termed the primal problem. The dual is then obtained by imposing (5.11, 12, 13, 14) on to V^* of (5.10) to obtain, as will be seen, a maximum problem

$$\max W = (1/\sigma\epsilon) \left\{ \iint_R (Xu + Yv)\,dx\,dy + \int_{B_F} (X_n u + Y_n v)\,ds \right\}. \qquad (5.15)$$

This depends upon the dual 'variables' u, v only and so, in formulating the dual constraints, the primal 'variables' t, ϕ, T_1, T_2, p_i and $q_i (i = 1, 2, 3)$ should be eliminated from (5.11, 12, 13, 14). This gives, taking $\gamma = 0$ everywhere,

$$\left.\begin{aligned} &\sum_{i=1}^{3} (\lambda_i + \mu_i) \leqslant \epsilon, \quad \lambda_1 - \mu_1 + \lambda_3 - \mu_3 = \epsilon_1{}^*, \quad \lambda_2 - \mu_2 - \lambda_3 + \mu_3 = \epsilon_2{}^*, \\ &\lambda_i, \mu_i \geqslant 0 \quad (i = 1, 2, 3), \quad u = v = 0 \text{ on } B_C, \quad u, v \text{ continuous in } R, \end{aligned}\right\} \qquad (5.16)$$

where $\epsilon_1{}^*$ and $\epsilon_2{}^*$ are the quantities, which are obtained from (5.12) by writing

$\gamma = 0$ and eliminating ϕ, i.e. the principal strains corresponding to the displacements u, v. The dual problem is thus to maximize W of (5.15), subject to the constraints of (5.16).

Let T_1, T_2, ϕ, T_x, T_y, S, t, p_i and $q_i (i = 1, 2, 3)$ be any solution of the primal constraints (5.1, 2, 3, 4, 5, 6) and let u, v, λ_i and $\mu_i (i = 1, 2, 3)$† be any solution of the dual constraints (5.16), then, beginning with (5.15),

$$W = (1/\sigma\epsilon)\left[\iint_R (Xu + Yv)\,dx\,dy + \int_{B_F + B_C} \{u(lT_x + mS) + v(lS + mT_y)\}\,ds\right], \quad \text{by (5.3, 16)}$$

$$= (1/\sigma\epsilon)\iint_R \{\partial(uT_x + vS)/\partial x + \partial(uS + vT_y)/\partial y + Xu + Yv\}\,dx\,dy, \quad \text{by Green's theorem and (5.4)},$$

$$= (1/\sigma\epsilon)\iint_R \{T_x\,\partial u/\partial x + T_y\,\partial v/\partial y + S(\partial v/\partial x + \partial u/\partial y)\}\,dx\,dy, \quad \text{by (5.2)},$$

$$= (1/\sigma\epsilon)\iint_R (T_1{}^*\epsilon_1{}^* + T_2{}^*\epsilon_2{}^*)\,dx\,dy, \tag{5.17}$$

where $T_1{}^*$, $T_2{}^*$ and S^* are the stress resultants referred to the principal directions of strain, for which $(\epsilon_1{}^*, \epsilon_2{}^*, 0)$ are the corresponding components of strain. The invariance of $T_x\,\partial u/\partial x + T_y\,\partial v/\partial y + S(\partial v/\partial x + \partial u/\partial y)$ has been used in the last line of (5.17).

Tresca's conditions of (5.5, 6) imply that all shear stresses in the plate are not greater than $\sigma/2$. It thus follows that $|T_1{}^*|/2t$, $|T_2{}^*|/2t$ and $[\{(T_1{}^* - T_2{}^*)/2\}^2 + S^{*2}]^{1/2}/t$ cannot exceed this stress. Slack functions $p_i{}^*$, $q_i{}^*$ $(i = 1, 2, 3)$ can thus be found such that

$$\left.\begin{aligned} T_1{}^* + p_1{}^* &= \sigma t, & -T_1{}^* + q_1{}^* &= \sigma t,\\ T_2{}^* + p_2{}^* &= \sigma t, & -T_2{}^* + q_2{}^* &= \sigma t,\\ T_1{}^* - T_2{}^* + p_3{}^* &= \sigma t, & -T_1{}^* + T_2{}^* + q_3{}^* &= \sigma t,\\ \text{with} \quad & p_i{}^*,\ q_i{}^* \geqslant 0 & (i &= 1, 2, 3). \end{aligned}\right\} \tag{5.18}$$

Returning to (5.17) and substituting from (5.16) gives

$$W = (1/\sigma\epsilon)\iint_R \{T_1{}^*(\lambda_1 - \mu_1 + \lambda_3 - \mu_3) + T_2{}^*(\lambda_2 - \mu_2 - \lambda_3 + \mu_3)\}\,dx\,dy,$$

$$= (1/\sigma\epsilon)\iint_R \{\lambda_1 T_1{}^* + \mu_1(-T_1{}^*) + \lambda_2 T_2{}^* + \mu_2(-T_2{}^*) + \lambda_3(T_1{}^* - T_2{}^*) + \mu_3(-T_1{}^* + T_2{}^*)\}\,dx\,dy,$$

$$= (1/\sigma\epsilon)\iint_R \{\sigma t\sum_{i=1}^{3}(\lambda_i + \mu_i) - \sum_{i=1}^{3}(\lambda_i p_i{}^* + \mu_i q_i{}^*)\}\,dx\,dy, \quad \text{by (5.18)},$$

$$\leqslant \iint_R t\,dx\,dy, \quad \text{by (5.16, 18)},$$

$$= V, \quad \text{by (5.7)}. \tag{5.19}$$

It follows by (5.19) that W is bounded above and that, if the primal has a solution $V_{\min}$, then

$$W \leqslant V_{\min} \leqslant V, \tag{5.20}$$

and so feasible solutions to the primal and dual constraints give upper and lower bounds to $V_{\min}$. If these solutions also satisfy (5.11), which means that

$$\left.\begin{aligned} &t\sum_{i=1}^{3}(\lambda_i + \mu_i) = t\epsilon,\\ &\epsilon_1 = \epsilon_1{}^*, \quad \epsilon_2 = \epsilon_2{}^*, \quad (T_1 - T_2)\gamma = 0,\\ &\lambda_i p_i = 0, \quad \mu_i q_i = 0 \quad (i = 1, 2, 3), \end{aligned}\right\} \tag{5.21}$$

it follows that the principal directions of strain coincide with the principal directions of the stress resultants or, in the case of isotropic strain, can be taken to coincide and so $T_1{}^* = T_1$ and $T_2{}^* = T_2$. Equations (5.5, 18) then give $p_i{}^* = p_i$ and $q_i{}^* = q_i$ $(i = 1, 2, 3)$. Reference to the third step of (5.19) and to (5.21) shows that $W = V$, which, by (5.20), must equal $V_{\min}$. The dual and the primal both have solutions under the present assumptions and

$$W_{\max} = V_{\min}. \tag{5.22}$$

The present considerations lead to sufficient conditions for an optimum design. They are expressed mathematically by (5.1–6), i.e. the constraints of the primal, and (5.11–14), with the understanding that $\gamma = 0$ everywhere, even when $T_1 = T_2$. If the theorem is true that the necessary conditions for the Lagrangian V^* of (5.8) to be a minimum also give the necessary conditions that V of (5.7) should be a minimum, when subjected to the constraints (5.1–6), then the conditions (5.1–6, 11–14) are necessary as well as sufficient for an optimum design.

5.4 Nature of the optimum solution

Much progress can often be made using sufficient conditions, as Chapter 4 has shown, and the same is true here, since the present solutions have much in common with those for Michell continua.

The optimum solution cannot have a finite region in which $p_i > 0$, $q_i > 0$ $(i = 1, 2, 3)$. Equation (5.5) shows that $t > 0$, while (5.11) gives $\lambda_i = \mu_i = 0$ $(i = 1, 2, 3)$ and hence $t = 0$.

There are six possible cases for which one of p_i, q_i $(i = 1, 2, 3)$ is equal to zero, while the rest are positive. Equation (5.11) then implies that all but one of λ_i, μ_i $(i = 1, 2, 3)$ are zero and that the positive one is equal to ϵ, since $t > 0$ for this case. The principal strains ϵ_1, ϵ_2 are also constant, and take the values $\pm\epsilon$ or 0. In detail (5.5, 11) give:

$$\left.\begin{array}{ll}
\text{(a)} & T_1 = \sigma t,\ \ 0 < T_2 < \sigma t,\ \ \epsilon_1 = \epsilon,\ \ \epsilon_2 = 0,\\
\text{(b)} & T_1 = -\sigma t,\ \ -\sigma t < T_2 < 0,\ \ \epsilon_1 = -\epsilon,\ \ \epsilon_2 = 0,\\
\text{(c)} & T_2 = \sigma t,\ \ 0 < T_1 < \sigma t,\ \ \epsilon_1 = 0,\ \ \epsilon_2 = \epsilon,\\
\text{(d)} & T_2 = -\sigma t,\ \ -\sigma t < T_1 < 0,\ \ \epsilon_1 = 0,\ \ \epsilon_2 = -\epsilon,\\
\text{(e)} & T_1 - T_2 = \sigma t,\ \ 0 < T_1 < \sigma t,\ \ -\sigma t < T_2 < 0,\ \ \epsilon_1 = \epsilon,\ \ \epsilon_2 = -\epsilon,\\
\text{(f)} & T_1 - T_2 = -\sigma t,\ \ -\sigma t < T_1 < 0,\ \ 0 < T_2 < \sigma t,\ \ \epsilon_1 = -\epsilon,\ \ \epsilon_2 = \epsilon.
\end{array}\right\} \tag{5.23}$$

There are six further cases that can be obtained by setting two of p_i, q_i $(i = 1, 2, 3)$ equal to zero, while the rest are positive. These correspond to the corners of the hexagon of Fig. 3.2. Equations (5.5, 11) give for these:

$$\left.\begin{array}{ll}
\text{(g)} & T_1 = \sigma t > 0,\ \ T_2 = 0,\ \ \epsilon_1 = \epsilon,\ \ -\epsilon \leqslant \epsilon_2 \leqslant 0,\\
\text{(h)} & T_1 = -\sigma t < 0,\ \ T_2 = 0,\ \ \epsilon_1 = -\epsilon,\ \ 0 \leqslant \epsilon_2 \leqslant \epsilon,\\
\text{(i)} & T_1 = 0,\ \ T_2 = \sigma t > 0,\ \ -\epsilon \leqslant \epsilon_1 \leqslant 0,\ \ \epsilon_2 = \epsilon,\\
\text{(j)} & T_1 = 0,\ \ T_2 = -\sigma t < 0,\ \ 0 \leqslant \epsilon_1 \leqslant \epsilon,\ \ \epsilon_2 = -\epsilon,\\
\text{(k)} & T_1 = T_2 = \sigma t > 0,\ \ \epsilon_1 + \epsilon_2 = \epsilon,\ \ \epsilon_1 \geqslant 0,\ \ \epsilon_2 \geqslant 0,\\
\text{(l)} & T_1 = T_2 = -\sigma t < 0,\ \ \epsilon_1 + \epsilon_2 = -\epsilon,\ \ \epsilon_1 \leqslant 0,\ \ \epsilon_2 \leqslant 0.
\end{array}\right\} \tag{5.24}$$

The degenerate case

$$\text{(m)}\ \ T_1 = T_2 = t = 0,\ |\epsilon_1| + |\epsilon_2| + |\epsilon_1 + \epsilon_2| \leqslant 2\epsilon \tag{5.25}$$

must be recorded for completeness. The stated restriction on strain can be derived from (5.11), remembering that $p_i = q_i = 0$ $(i = 1, 2, 3)$ for this case.

All regions of an optimum solution must satisfy

$$T_1\epsilon_1 + T_2\epsilon_2 = \sigma\epsilon t. \tag{5.26}$$

This follows from (5.23–25) and was proved, in passing, when (5.22) was established. If ϵ_1, ϵ_2 are thought of as 'rates of strain', then since the states of stress corresponding to T_1 and T_2 in (5.23–25) all permit yielding, equation (5.26) means that the dissipation per unit volume is a constant (Drucker and Shield, 1956). The strains of (5.23, 24) satisfy the flow rules of plasticity theory, since the vector (ϵ_1, ϵ_2) is normal to the corresponding side of the 'yield hexagon' of Fig. 3.2 in the case of (5.23) and lies between the normals to adjacent sides in (5.24).

A complete optimum solution will consist of a finite number of regions of types (a) to (m) completely filling the region R. Condition (5.3) must be satisfied on the boundary B_F, condition (5.4) at the boundaries between regions, condition (5.13) on B_C and condition (5.14) everywhere. Suitable forms for T_1, T_2, ϕ, t, u and v must be obtained by integration of (5.1, 2) and (5.12), with $\gamma = 0$, together with the relevant equations from (5.23, 24).† The disposable functions thus obtained must be found from the boundary conditions listed above. In each region the inequalities of (5.23, 24) and those implied for (m) of (5.25) must be satisfied, before the solution can be regarded as complete. The difficulty of the present problem lies in fact that the various regions and the type of solution applicable to each cannot be determined in advance. The only method is to try appropriate solutions, which are possible at the various parts of the boundary and attempt to fit them together, with perhaps further solutions to completely fill the region R. A detailed knowledge of known solutions of the various types is an obvious prerequisite for this process.

5.5 Virtual-strain fields

It is convenient here, as in Section 4.2, to use the lines of principal virtual strain as coordinate lines for a system of curvilinear coordinates (α, β). The purely geometric developments of equations (4.11–16) are thus available for application to the present problem.

Equations (4.17) require modification, so as to deal with the slightly more general strain conditions of (5.23). It is sufficient to replace $\sigma\epsilon/\sigma_T$ by ϵ_1 and $\sigma\epsilon/\sigma_C$ by ϵ_2, where ϵ_1 and ϵ_2 are constants, which must be given the values required by (5.23) for regions of the plate satisfying one of the specifications (a) to (f). Equations (4.19, 21) then become

$$u + iv = e^{i(\phi_0 - \phi)}(u_0 + iv_0) + e^{i\phi}\int_{(0,0)}^{(\alpha,\beta)} e^{i\phi}\{A\epsilon_1\,d\alpha + iB\epsilon_2\,d\beta + i\omega\,(A\,d\alpha + iB\,d\beta)\}, \tag{5.27}$$

and

$$\omega = \omega_0 + (\epsilon_1 - \epsilon_2)\int_{(0,0)}^{(\alpha,\beta)} \{(\partial\phi/\partial\alpha)\,d\alpha - (\partial\phi/\partial\beta)\,d\beta\}, \tag{5.28}$$

† This is a general statement and must not be taken too literally. The analysis will usually be carried out in appropriate curvilinear coordinates (see Chapter 4).

where it should be noted, that here, as in Chapter 4, (u, v) are component displacements in the curvilinear coordinate system (along the coordinate lines) and not just Cartesian components, as in Sections 5.2–4 above.

Equations (4.22–24) follow from (5.28) just as they did from (4.21). This means that, for the cases governed by (5.23), the lines of principal strain form Hencky nets. The considerations of Section 4.4 and the many examples of Sections 4.5–9 are thus applicable to cases (a) to (f) of (5.23). In the special cases (e) and (f) even the displacements are given by the same formulae, when $\sigma_T = \sigma_C = \sigma$.

The boundary conditions on constrained parts of the boundary (5.13) give, by (5.27), a replacement for (4.25) in the form

$$\omega = \pm\sqrt{(-\epsilon_1\epsilon_2)}, \qquad A\sqrt{(\epsilon_1)}\,\mathrm{d}\alpha = \pm B\sqrt{(-\epsilon_2)}\mathrm{d}\beta \text{ on } B_C. \tag{5.29}$$

The cases governed by (g) to (j) of (5.24) as typified by (g), have been dealt with already in equations (4.26–31), under the special assumption, which is all that will be considered here, that the α-lines are straight. The only modifications required are to write $\sigma_T = \sigma$ and to replace (4.31) by one of the rather more restrictive inequality systems of (5.24).

The kinematic conditions of (k) and (l) of (5.24) impose no restrictions on the layout of the principal strain lines. Also since $T_1 = T_2$ no restriction arises from the statics either. This means that for simplicity it is permissible to take (α, β) in (k) or (l) as rectangular Cartesian coordinates, with $A = B = 1$ and $\phi = \phi_0$. The conditions for (k) then give

$$\partial u/\partial\alpha + \partial v/\partial\beta = \epsilon, \qquad \partial v/\partial\alpha + \partial u/\partial\beta = 0, \tag{5.30}$$

which integrate to give

$$u = \epsilon\alpha/2 + F_1(\alpha+\beta) + F_2(\alpha-\beta), \qquad v = \epsilon\beta/2 - F_1(\alpha+\beta) + F_2(\alpha-\beta), \tag{5.31}$$

where F_1 and F_2 are arbitrary functions. Equation (5.24) requires $\epsilon_1, \epsilon_2 \geqslant 0$ and so F_1 and F_2 must satisfy

$$-\epsilon/2 \leqslant F_1'(\alpha+\beta) + F_2'(\alpha-\beta) \leqslant \epsilon/2. \tag{5.32}$$

The case (m) of (5.25) is even less determinate, since only inequalities are imposed. The strain field for this degenerate case can of course be taken as a strain field for a non-degenerate case, if this will give a continuous deformation.

5.6 Conditions of equilibrium

The equations of equilibrium (4.33) may be applied to the present problem, when body forces (X, Y) are absent, by writing

$$\mathbf{T}_1 = BT_1, \qquad \mathbf{T}_2 = AT_2. \tag{5.33}$$

Equations (4.34, 35) may be applied here as boundary conditions, with or without the concentrated members. Concentrated forces may be conveniently dealt with by introducing concentrated members, rather than singularities of plate thickness.

The required plate thicknesses may be calculated from (5.23, 24). Introducing $\mathbf{T}_1$ and $\mathbf{T}_2$ from (5.33) gives

$$\left.\begin{array}{ll} \text{cases (a), (b), (g) and (h)} & t = |\mathbf{T}_1|/\sigma B, \\ \text{cases (c), (d), (i) and (j)} & t = |\mathbf{T}_2|/\sigma A, \\ \text{cases (e) and (f)} & t = |\mathbf{T}_1/B - \mathbf{T}_2/A|/\sigma. \end{array}\right\} \tag{5.34}$$

It is to be remarked that case (e) agrees with (4.36), with $\sigma_T = \sigma_C = \sigma$, and that case (g) agrees with (4.37), with $\sigma_T = \sigma$, since, by (4.26) $B = \alpha + F(\beta)$. It follows, as should be expected, for cases (e), (f), (g), (h), (i) and (j), that the 'equivalent thickness' of a Michell continuum is equal to the thickness of the optimum plate, when the 'loading' T_1 and T_2 is the same. Cases (a), (b), (c) and (d) do not arise for Michell continua. The plate has the advantage that, when stressed to the limit σ in one sense, it can carry stresses of the same sign in a direction at right angles up to the limit σ, without any additional material being required.

The fact that, for cases (g), (h), (i) and (j), equilibrium, in the absence of body forces, requires one set of coordinate lines to be straight, follows from (4.33) as before.

Application of (4.33), (5.33) and (4.14) to (k) and (l) of (5.24) gives

$$\text{cases (k) and (l)} \qquad t = |T_1|/\sigma = |T_2|/\sigma = \text{constant} \tag{5.35}$$

which is to be expected, since a hydrostatic system of stress cannot vary in the absence of body forces. The magnitude of t in (5.35) is determined by the boundary conditions. The Michell continuum is not to be recommended for cases (k) and (l). For $T_1 = T_2$, $t_1 = t_2 = |T_1|/\sigma_T$ and so the equivalent thickness is $2|T_1|/\sigma_T$ or twice the t of (5.35), when $\sigma_T = \sigma$. The advantage of plates for like signed principal stresses has been noticed above.†

Finally it may be remarked that integrals of the equilibrium equations (4.33), valid for special coordinate systems, like (4.41) and special forms of those equations, like (4.52), can be applied to plate problems.

5.7 Examples

It is now clear that any Michell continuum, which is based on a strain field $\epsilon_1 = \sigma\epsilon/\sigma_T$, $\epsilon_2 = -\sigma\epsilon/\sigma_C$ and which has a stress distribution for which $\mathbf{T}_1$ and

† This point was made in Richards and Chan (1966).

T_2 have corresponding signs, can be converted into an optimum plate for the same loading system by writing $\sigma_T = \sigma_C = \sigma$. Such examples give the case (e) of (5.23) or (g) of (5.24), with in the last case ϵ_2 at its lower limit $-\epsilon$. Cases (f) of (5.23) and (h), (i) and (j) of (5.24) can be realized by changing the sign of the loads and/or interchanging the role of α and β.

Fig. 4.5 may be interpreted as a plate bounded by a concentrated member, with the layout lines giving lines of principal stress. The thickness t is equal to $F/\sigma r$ and by (4.56) $V_{\min} = 2\pi lF/\sigma$.

The strain field of Fig. 4.6 is not valid for an optimum plate, since $\epsilon_1 = \epsilon_2 = -\epsilon$ contradicts (5.24) (l) and indeed all the strain conditions of (5.23–25). However, if $\theta_0 = \pi/4$, the region of hydrostatic strain disappears and the resulting field of Fig. 4.8 is satisfactory. Fig. 4.7(a) gives an optimum design in which the fan may be replaced by a plate and the same may be stated for the designs of Fig. 4.9. The volume for cases (a) and (b) of Fig. 4.9 is by (4.70) given by $V_{\min} = (\pi + 2)Fl/\sigma$.

The field of Fig. 4.11 is a valid strain field and the design of Fig. 4.12 may be interpreted as a plate with edge members. The required plate thickness for the optimum is given by (4.92) and the total volume by (4.93). The design of Fig. 4.13 for the transmission of moments can also be interpreted as a plate design. It is interesting to note that region (b) has $\epsilon_1 = \epsilon_2 = 0$ and thus provides an example of (5.25). The volume of material for this last is given by (4.95). Fig. 4.14 also provides an example of a pure plate design based on the field of Fig. 4.11.

The field of cycloids of Fig. 4.15 is also satisfactory for an optimum plate and the design of Fig. 4.16 with a web of thickness $t = t_1 + t_2$, as given by (4.109), is much more practical than a spider's web of cycloids. Equation (4.114) is valid for the plate design.

The designs of Figs. 4.18, 4.20, and 4.21 and of others like them can all be regarded as plate designs, with the same volumes of material as that given, in for example (4.123), for the corresponding Michell continua.†

The almost trivial example of a region R of any form, simply or multiply connected, loaded by a uniform normal pressure along its boundary, may also be noted. This is solved by (5.24) (l) with (say) $\epsilon_1 = \epsilon_2 = -\epsilon/2$. Equation (5.35) shows that the plate thickness is uniform in the optimum design for this loading case.

The examples given above are derived from a theory of optimum plastic design. They also provide examples of optimum elastic design if their virtual strain system can be their actual strain system. This is a severe requirement and is only met by two of the examples, namely that of Fig. 4.14 for the transmission of torque and the uniform plate under uniform pressure at its edges.

† Numerical examples of the designs of Figs. 4.18 and 4.12 are given in Richards and Chan (1966).

Sufficient conditions can be derived from (5.23, 24) in the form:

$$\left.\begin{array}{ll}
\text{(a)} & T_1/t = \sigma,\ T_2/t = \nu\sigma,\ \epsilon = \sigma(1-\nu^2)/E, \\
\text{(e)} & T_1/t = -T_2/t = \sigma/2,\ \epsilon = \sigma(1+\nu)/2E, \\
\text{(g)} & T_1/t = \sigma,\ T_2/t = 0,\ \epsilon = \sigma/E,\ \epsilon_2 = -\nu\epsilon, \\
\text{(k)} & T_1/t = T_2/t = \sigma,\ \epsilon_1 = \epsilon_2 = \sigma(1-\nu)/E,\ \epsilon = 2\sigma(1-\nu)/E,
\end{array}\right\} \tag{5.36}$$

where E is Young's modulus and ν is Poisson's ratio. The principal stresses are constant in each region. The structure for torque is an example of (e) and that for pressure of (k), with its signs reversed, i.e. of (l).

5.8 Optimum design with the Mises–Hencky criterion

If the Mises–Hencky criterion is adopted for defining the condition for the safety of the plate, then (5.5, 6) must be replaced by

$$(T_x^2 + T_y^2 - T_xT_y + 3S^2)^{1/2} + p = \sigma t, \qquad p \geqslant 0. \tag{5.37}$$

The Lagrangian of (5.8) then becomes

$$\begin{aligned}
V^* = (1/\sigma\epsilon)\iint_R & [\sigma\epsilon t + \lambda\{(T_x^2 + T_y^2 - T_xT_y + 3S^2)^{1/2} + p - \sigma t\} + \\
& + u(\partial T_x/\partial x + \partial S/\partial y + X) + v(\partial S/\partial x + \partial T_y/\partial y + Y)]\ \mathrm{d}x\,\mathrm{d}y - \\
& - (1/\sigma\epsilon)\int_{B_F} \{u_1(lT_x + mS - X_n) + v_1(lS + mT_y - Y_n)\}\,\mathrm{d}s.
\end{aligned} \tag{5.38}$$

The transformation of (5.9) can be applied once more to yield, instead of (5.10), the equation

$$\begin{aligned}
V^* = (1/\sigma\epsilon)\iint_R & [\sigma\epsilon t + \lambda\{(T_x^2 + T_y^2 - T_xT_y + 3S^2)^{1/2} + p - \sigma t\} - \\
& - T_x\epsilon_x - T_y\epsilon_y - S\gamma + uX + vY]\ \mathrm{d}x\,\mathrm{d}y + \\
& + (1/\sigma\epsilon)\int_{B_F} \{(u - u_1)(lT_x + mS) + (v - v_1)(lS + mT_y) + u_1X_n + v_1Y_n\}\,\mathrm{d}s + \\
& + (1/\sigma\epsilon)\int_{B_C} \{u(lT_x + mS) + v(lS + mT_y)\}\,\mathrm{d}s + \\
& + (1/\sigma\epsilon)\int_{D} \{\Delta u(lT_x + mS) + \Delta v(lS + mT_y)\}\,\mathrm{d}s,
\end{aligned} \tag{5.39}$$

where,

$$\epsilon_x = \partial u/\partial x, \quad \epsilon_y = \partial v/\partial y, \quad \gamma = \partial v/\partial x + \partial u/\partial y, \tag{5.40}$$

and if (u, v) is taken as a virtual displacement in the coordinate system (x, y) then (5.40) defines the usual strain components.

The conditions for V^* to be a minimum can now be written as

$$\lambda = \epsilon \quad (t>0), \qquad \lambda \leqslant \epsilon \quad (t=0), \tag{5.41}$$

$$\left.\begin{aligned} \epsilon_x &= \lambda(T_x - T_y/2)/(T_x^2 + T_y^2 - T_xT_y + 3S^2)^{1/2}, \\ \epsilon_y &= \lambda(T_y - T_x/2)/(T_x^2 + T_y^2 - T_xT_y + 3S^2)^{1/2}, \\ \gamma &= 3\lambda S/(T_x^2 + T_y^2 - T_xT_y + 3S^2)^{1/2}, \end{aligned}\right\} \tag{5.42}$$

$$\lambda = 0 \quad (p>0), \qquad \lambda \geqslant 0 \quad (p=0), \tag{5.43}$$

$$u_1 = u, \quad v_1 = v \text{ on } B_F, \quad u = v = 0 \text{ on } B_C \text{ and } u, v \text{ continuous.} \tag{5.44}$$

Equations (5.42) imply that

$$\epsilon_x^2 + \epsilon_y^2 + \epsilon_x\epsilon_y + \gamma^2/4 = 3\lambda^2/4 \tag{5.45}$$

and

$$T_x\epsilon_x + T_y\epsilon_y + S\gamma = \lambda(T_x^2 + T_y^2 - T_xT_y + 3S^2)^{1/2}. \tag{5.46}$$

Equations (5.42) are the flow rules for perfect plastic flow, if ϵ_x, ϵ_y and γ are taken as rates of strain and so the virtual deformation can be thought of as a collapse mechanism. The dissipation per unit area is given by (5.46), with $\lambda = \epsilon$ by (5.41) and the square root replaced by σt from (5.37) with $p = 0$, which is required by (5.43). The dissipation per unit volume is thus a constant $\sigma\epsilon$ as is required by the condition of Drucker and Shield (1956).

The dual problem can be obtained by imposing (5.41–44) on (5.39). This gives using (5.46)

$$\max W = (1/\sigma\epsilon)\left\{\iint_R (Xu + Yv)\,dx\,dy + \int_{B_F} (X_n u + Y_n v)\,ds\right\}, \tag{5.47}$$

subject to

$$\left.\begin{aligned} &(\epsilon_x^2 + \epsilon_y^2 + \epsilon_x\epsilon_y + \gamma^2/4)^{1/2} \leqslant \sqrt{3}\epsilon/2, \\ &u, v \text{ continuous and } u = v = 0 \text{ on } B_C, \end{aligned}\right\} \tag{5.48}$$

where (5.45) has been used to eliminate the stress resultants.

The usual inequality between W and V, when calculated from solutions of the dual and primal constraints, can now be established. A beginning is made with

$$W = (1/\sigma\epsilon) \iint_R (T_x\epsilon_x + T_y\epsilon_y + S\gamma)\,dx\,dy, \tag{5.49}$$

which follows from (5.17, 40). The next step requires the inequality

$$T_x\epsilon_x + T_y\epsilon_y + S\gamma \leqslant (2/\sqrt{3})\{(T_x^2 + T_y^2 - T_xT_y + 3S^2)(\epsilon_x^2 + \epsilon_y^2 + \epsilon_x\epsilon_y + \gamma^2/4)\}^{1/2}, \tag{5.50}$$

which is best established by introducing the 'deviatoric components' of the stress resultant and strains

$$\left.\begin{aligned} &T_x' = 2T_x/3 - T_y/3, \quad T_y' = 2T_y/3 - T_x/3, \quad T_z' = -T_x/3 - T_y/3, \quad S' = S, \\ &\epsilon_x' = \epsilon_x, \quad \epsilon_y' = \epsilon_y, \quad \epsilon_z' = -\epsilon_x - \epsilon_y, \quad \gamma' = \gamma. \end{aligned}\right\} \tag{5.51}$$

Equations (5.51) imply

$$\left.\begin{aligned} &T_x'\epsilon_x' + T_y'\epsilon_y' + T_z'\epsilon_z' + 2S'(\gamma'/2) = T_x\epsilon_x + T_y\epsilon_y + S\gamma, \\ &T_x'^2 + T_y'^2 + T_z'^2 + 2S'^2 = (2/3)(T_x^2 + T_y^2 - T_xT_y + 3S^2), \\ &\epsilon_x'^2 + \epsilon_y'^2 + \epsilon_z'^2 + 2(\gamma'/2)^2 = 2(\epsilon_x^2 + \epsilon_y^2 + \epsilon_x\epsilon_y + \gamma^2/4). \end{aligned}\right\} \tag{5.52}$$

Application of Cauchy's inequality† to the left hand sides of the equations of (5.52), then leads by (5.52) to (5.50). Finally use of (5.50), (5.37) and (5.48) gives by (5.49)

$$W \leqslant (1/\sigma\epsilon) \iint_R (2/\sqrt{3})\,\sigma t(\sqrt{3}\epsilon/2)\,dx\,dy = V. \tag{5.53}$$

In the special case when the solutions used to calculate W and V in (5.53) satisfy (5.41–44), it follows by (5.49), (5.46) and (5.37) that

$$W = (1/\sigma\epsilon) \iint_R \lambda(\sigma t - p)\,dx\,dy = V, \tag{5.54}$$

on using (5.41, 43). All the usual deductions can now be made.

The solution of problems using the equations derived in this section is clearly not an easy matter. However, progress can be made by using (5.53) to establish bounds to an exact solution. For example let it be supposed that a solution to a problem using Tresca's criterion is known and let the minimum volume be $V_{\min}^{\mathrm{T}}$. If a system of stress satisfies (5.5, 6) then it must satisfy (5.37). This follows from the graphical representation of (5.37), which gives as a boundary the circumscribed ellipse of the Tresca hexagon of Fig. 3.2. It follows that $V_{\min}^{\mathrm{T}}$ is an upper bound to the minimum volume $V_{\min}^{\mathrm{M/H}}$ for the Mises–Hencky criterion. Comparison of the constraints of (5.16), or better the equivalent result of (5.25), with (5.48) can also be made graphically. Equation (5.25) gives an hexagonal boundary, which can be inscribed in the ellipse, obtained from (5.48) by transforming to principal directions,‡ by replacing ϵ in (5.25) by $\sqrt{3}\epsilon/2$. This reduces the virtual work of the external forces for the Tresca solution by a factor $\sqrt{3}/2$. Dividing this reduced virtual work, whose corresponding strains satisfy (5.48), by $\sigma\epsilon$ gives the lower bound $(\sqrt{3}/2)\,V_{\min}^{\mathrm{T}}$. Collecting

† The scalar product of two vectors is less than or equal to the product of their magnitudes, which is true for any number of dimensions.

‡ The strain condition of (5.48) is an invariant of the strain components.

these results gives

$$(\sqrt{3}/2)\, V^{\mathrm{T}}_{\min} \leqslant V^{\mathrm{M/H}}_{\min} \leqslant V^{\mathrm{T}}_{\min}, \tag{5.55}$$

which determines $V^{\mathrm{M/H}}_{\min}$ to within 7 per cent.

5.9 The problem of maximum stiffness

The general results obtained for other structures can be extended to plates loaded in their own plane, if the problem is formulated as that which seeks minimum stored strain energy for a given volume of structure. This can be written in the present case as

$$\left.\begin{aligned} &\min U = \iint_R [(1/2Et)\{(T_x + T_y)^2 + 2(1+\nu)(S^2 - T_x T_y)\}]\,\mathrm{d}x\,\mathrm{d}y,\\ &\text{subject to,}\\ &\iint_R t\,\mathrm{d}x\,\mathrm{d}y = V = \text{constant,} \end{aligned}\right\} \tag{5.56}$$

where E is Young's modulus and ν is Poisson's ratio.

A variation from t to $t + \delta t$ induces corresponding changes δT_x, etc., in the stress resultants due to the given external forces. These varied stress resultants are in equilibrium with the external forces and so the variations δT_x, etc., form an internal system of stress. It follows therefore that, if δT_x, etc., are imposed on U of (5.56), with t held unvaried, the resulting variation δU will be zero (Castigliano's principle).

Varying t in (5.56) thus gives

$$\left.\begin{aligned} &\iint_R [(-\delta t/2Et^2)\{(T_x + T_y)^2 + 2(1+\nu)(S^2 - T_x T_y)\}]\,\mathrm{d}x\,\mathrm{d}y \geqslant 0,\\ &\text{when}\\ &\iint_R \delta t\,\mathrm{d}x\,\mathrm{d}y = 0. \end{aligned}\right\} \tag{5.57}$$

In regions where $t > 0$, δt can have any sign and the inequality of (5.57) must be replaced by an equality. It then follows that

$$(1/2Et^2)\{(T_x + T_y)^2 + 2(1+\nu)(S^2 - T_x T_y)\} = \text{constant,} \qquad (t > 0) \tag{5.58}$$

or that the density of strain energy is uniformly distributed throughout the plate.

The result of (5.58) does not, of course, imply uniformity of stress distribution. However it is implied by it and so any solution, of Section 5.7, which has a uniform stress distribution and is also an elastic solution, gives an example of a

plate of maximum stiffness. The only examples which meet these requirements are those for the transmission of torque (Fig. 4.14) and for the transmission of pressure, using a uniform thickness, cited at the end of Section 5.7.

The optimum design of plates for given strength was found in Section 5.7 to be closely related to the corresponding problem of Michell continua. This is not the case for design for maximum stiffness. Michell continua can be treated like the frameworks of Section 1.12, but as has just been seen the problem for plates is strictly speaking a difficult non-linear problem in differential equations.

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